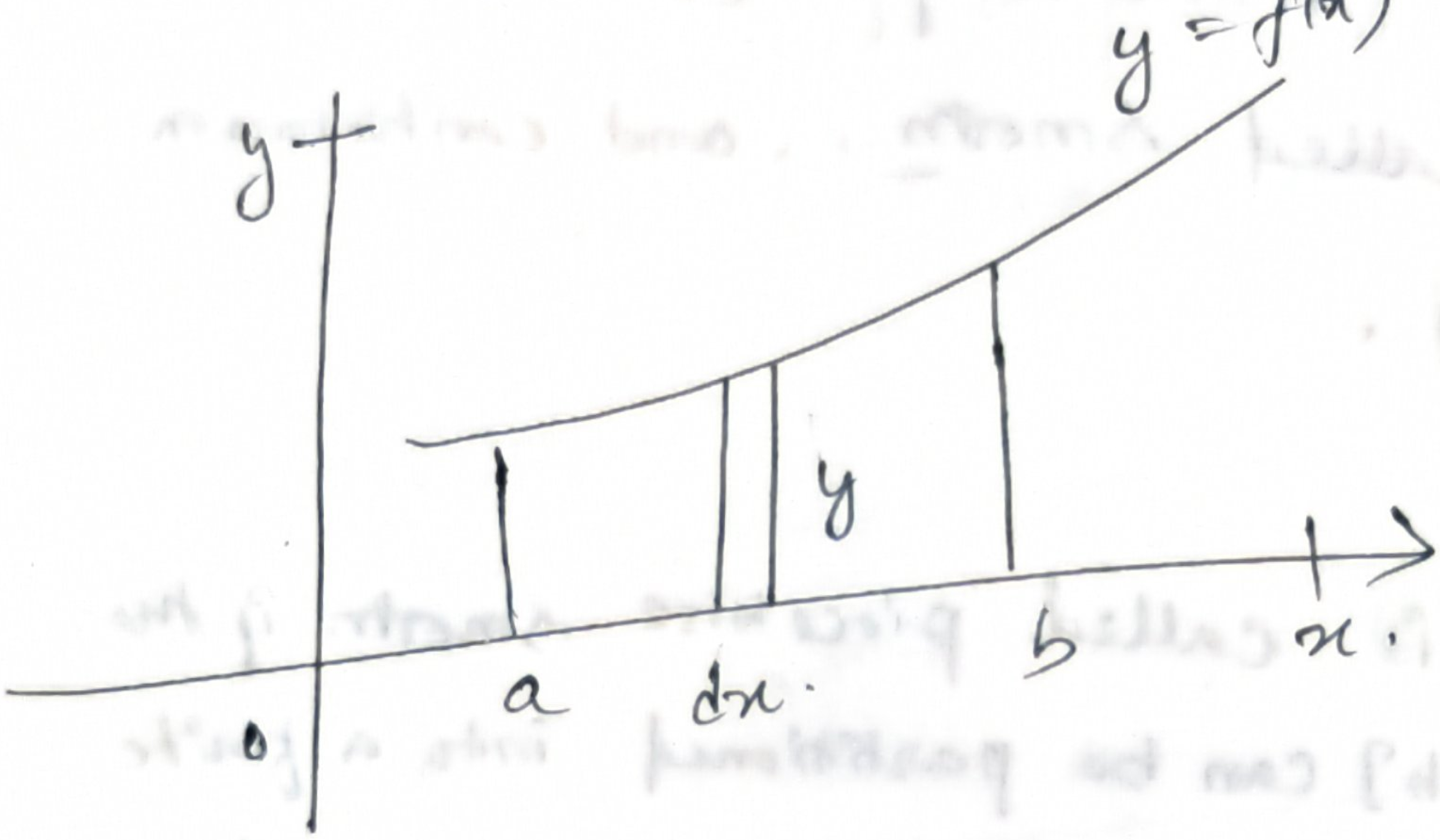
line integral.

⇒ For a line integral, the area is calculated under a curve defined by $y = f(x)$



Now to find the area of curve, one has to integrate $f(x)$ from a to b.

$$\boxed{\text{Area} = \int_a^b f(x)\,dx.}$$

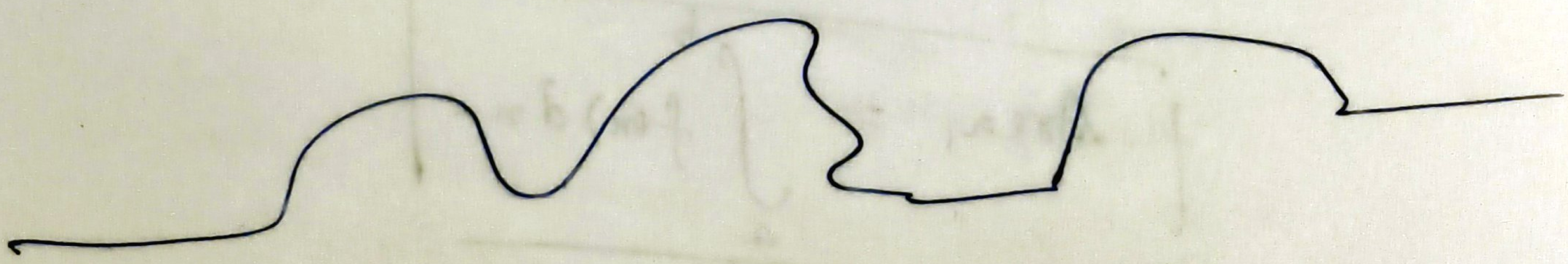
Given $I = [a, b]$

$f : [a, b] \longrightarrow \mathbb{R}$ which is continuous on

I is called a continuous path in n-space.

⇒ If f' i.e derivative f' exist, then such path is called smooth. and continuous on open (a, b).

⇒ The path is called piecewise smooth if the interval $[a, b]$ can be partitioned into a finite number of subinterval in each of which the path is smooth.



piecewise smooth path in the plane.

Let α be a piecewise smooth path in n-space defined on an interval $[a,b]$ and let f be a vector field defined and bounded on the graph of α. The line ~~Integ~~ integral of f along α is denoted by the symbol $\int f \cdot d\alpha$, and is defined by the equation

$$\boxed{\int f \cdot d\alpha = \int_a^b f[\alpha(t)] \cdot \alpha'(t)\,dt}$$

<u>Example</u>: let f be a two dimensional vector field given by $f(x,y) = \sqrt{y}\, i + (x^3+y)\, j$ for all (x,y), with $y \geq 0$.

Calculate the line integral of f from $(0,0)$ to $(1,1)$ along each of the following paths.

(a) the line with parametric equation

$$x = t,\ y = t,\ 0 \leq t \leq 1.$$

Soln:

$$\alpha(t) = it + tj$$

$$\frac{d\alpha(t)}{dt} = \alpha'(t) = i + j$$

$f(\alpha(t)) = f(it + tj)$

$= \sqrt{t}\, i + (t^3 + t) j.$

$$\int f \, d\alpha = \int_a^b f(\alpha(t)) \cdot \alpha'(t)\, dt.$$

$$= \int_a^b (\sqrt{t}\, i + (t^3 + t) j) \cdot (i + j) \cdot dt$$

Here $a = (0,0)$. $b = (1,1)$

$$= \int_{(0,0)}^{(1,1)} (\sqrt{t}\, i + (t^3 + t) j) \cdot (i + j)\, dt.$$

$$= \int_{(0,0)}^{(1,1)} f \cdot d\alpha = \int_0^1 (\sqrt{t} + t^3 + t)\, dt$$

$$= \frac{1}{2}\frac{1}{2\sqrt{t}}\Big]_0^1 + \frac{t^4}{4}\Big]_0^1 + \frac{t^2}{2}\Big]_0^1$$

$$= \frac{17}{12}.$$

Behaviour of a line integral :

Let α and β be equivalent piecewise smooth path, then we have

$$\int_C f \cdot d\alpha = \int_C f \cdot d\beta \quad \text{if } \alpha \text{ and } \beta \text{ trace out } C \text{ in same direction}$$

$$\int_C f \cdot d\alpha = -\int_C f \cdot d\beta \quad \text{if } \alpha \text{ and } \beta \text{ trace out } C \text{ in opposite direction.}$$

Q.1 ~~Calculat~~ Calculate the line integral of the vector field f along the path given by

$$f(x,y) = (x^2 - 2xy)\hat{i} + (y^2 - 2xy)\hat{j}$$

from $(-1,1)$ to $(1,1)$ along the parabola $y = x^2$

Soln:

$$f(x,y) = (x^2 - 2xy)\,i + (y^2 - 2xy)\,j$$

The parabola ~~lies~~ $(y = x^2)$ lies between the point $(-1,1)$ and $(1,1)$ can be denoted by $\alpha(t) = (t, t^2)$

where $t = x$.

i,e $\alpha(t) = (t, t^2)$ where $t \in [-1, 1]$.

$$\frac{d}{dt}\alpha(t) = \alpha'(t) = (1, 2t)$$
$$= 1i + 2tj$$

using the Formula

$$\boxed{\int f \cdot d\alpha = \int_a^b f[\alpha(t)] \cdot \alpha'(t)\,dt}$$

$$\int f \cdot d\alpha = \int_{-1}^{1} (t^2 - 2t \cdot t^2)i + (t^4 - 2t^3)j \cdot (1i + 2t)j\,dt.$$

we can write $f(x,y) = (x^2 - 2xy)i + (y^2 - 2xy)j$

$$f(t, t^2) = (t^2 - 2t^3, t^4 - 2t^3)$$

$$\int f \cdot d\alpha = \int_{-1}^{1} (t^2 - 2t^3, t^4 - 2t^3)(1, 2t)\,dt.$$

$$= \int_{-1}^{1} 1 \cdot (t^2 - 2t^3) + 2t(t^4 - 2t^3)\,dt$$

$$=$$

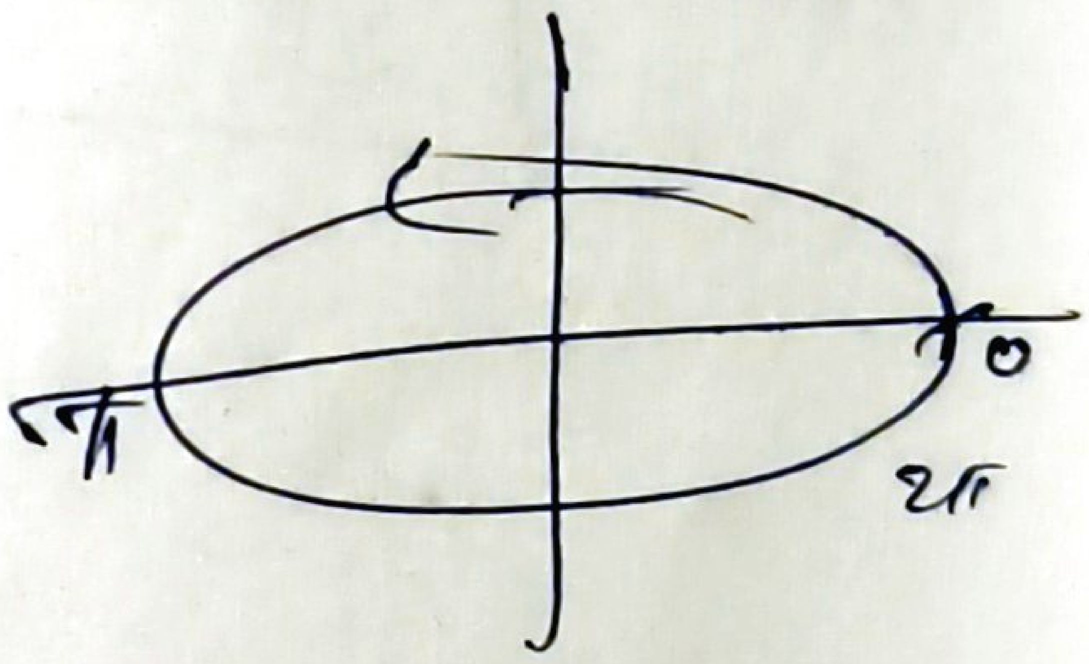
Q3 Calculate the line integral of the vector field f along the path given by $f(x,y) = (x+y)i + (x-y)j$, once around the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ in a counter clockwise direction.

Soln: $f(x,y) = (x+y)i + (x-y)j$

$f(x,y) = (x+y, x-y)$.

Now the path curve is denoted by.

$\alpha(t) = (a\cos t, b\sin t)$



where $t \in [0, 2\pi]$.

$\frac{d}{dt}(\alpha(t)) = \alpha'(t) = (-a\sin t, b\cos t)$

$$\boxed{\int f \cdot d\alpha = \int_0^{2\pi} f[\alpha(t)] \cdot \alpha'(t)\, dt}$$

$$= \int_0^{2\pi} (a\cos t + b\sin t, a\cos t - b\sin t) \cdot (-a\sin t, b\cos t)\, dt.$$

$$= \int_0^{2\pi} -a\sin t\,(a\cos t + b\sin t) + b\cos t\,(a\cos t - b\sin t)\,dt$$

$$= \int_0^{2\pi} (-a^2 - b^2)\cos t\sin t - ab\sin^2 t + ab\cos^2 t \; dt.$$

$$= \int_0^{2\pi} -a^2\cos t\sin t - b^2\sin t\cos t - ab\sin^2 t + ab\cos^2 t\,dt$$

Now take $a = b$. (constant).

then $\boxed{\iint f\,da = 0}$

Work of a line integral:

Let C be a smooth curve parametrized by $r(t)$, $a \le t \le b$ and F be a continuous ~~function~~ force field over a region containing C. Then the work done in moving an object from the point $A = r(a)$ to point $B = r(b)$ along C is

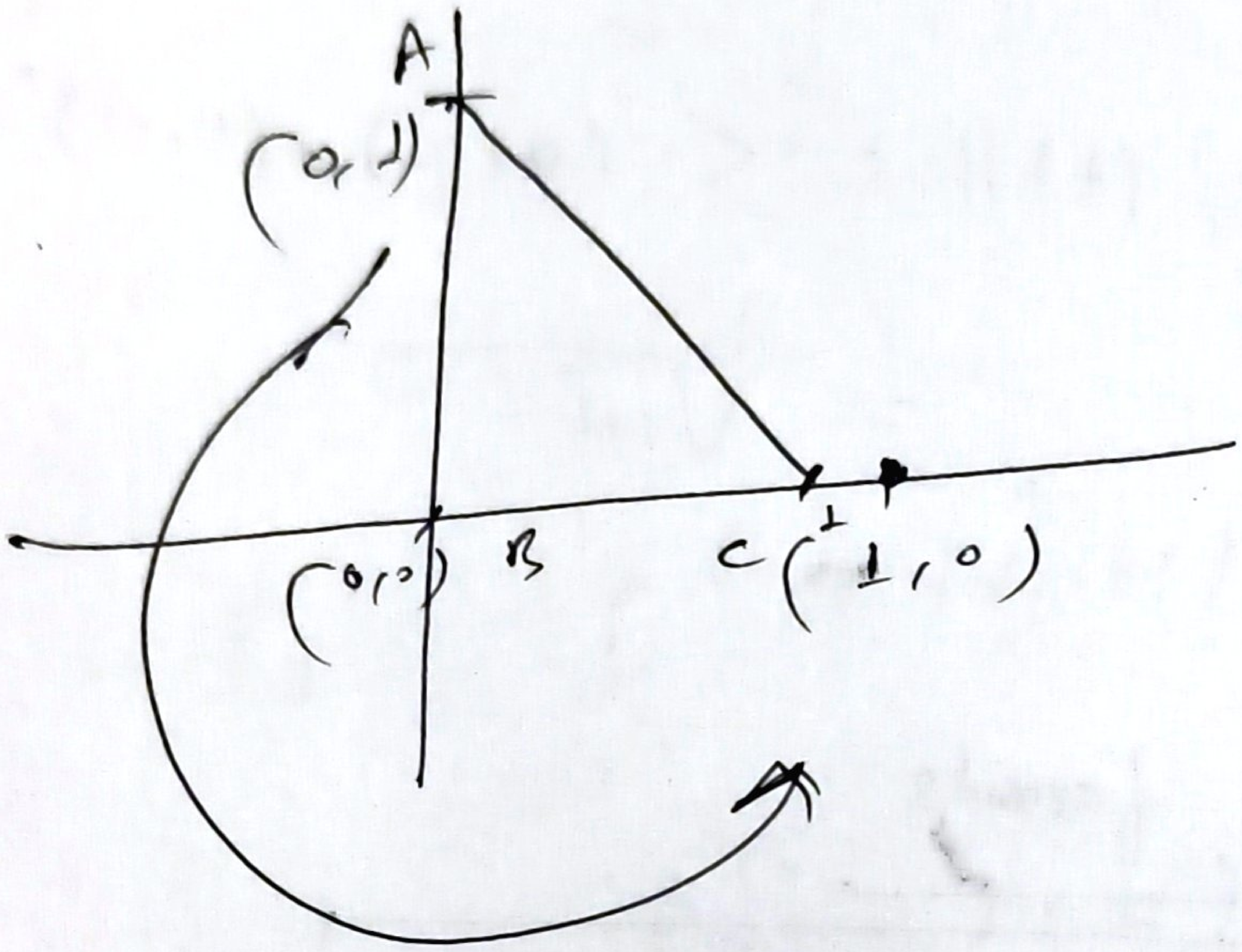
$$\text{Work} = \int_C F \cdot T \, ds$$

$$W = \int_C F \cdot T \, ds.$$

$$\boxed{\int_C f \, ds = \int_a^b f(r(t)) \, \| r'(t) \| \, dt.}$$

Example:- Calculate the line integral with respect to arc length given by $\int_C (x+y) \, ds$, where $F(x,y) = x+y$, and C is the triangle with vertices $(0,0)$, $(1,0)$ and $(0,1)$ traversed in a counterclockwise direction.

Soln. Given $F(x,y) = x + y$.



Start from A $(1,0)$

Now $\alpha_1(t) = (t, 0)$ where $t \in [0,1]$.

$\cancel{\alpha_1(t)}$ $\frac{d}{dt}\alpha_1 = (1,0)$.

$\|\alpha_1'(t)\| = 1$.

point B $(0,1)$

$\alpha_2(t) = (1-t, t)$ where $t \in [0,1]$

$\alpha_2'(t) = \frac{d}{dt}\alpha_2(t) = (-1, 1)$, where $t \in [0,1]$.

$\|\alpha_2'(t)\| = \langle(-1,1)\cdot(-1,1)\rangle^{1/2}$

$= \sqrt{2}$.

$\alpha_3(t) = (0, 1-t) \quad \forall\ t \in [0,1]$

$\frac{d}{dt}\alpha_3(t) = (0,-1)$

$\cancel{\|\alpha_3^t\|}$ $\quad \|\alpha_3'(t)\| = \langle (0,-1),(0,-1)\rangle^{1/2}$

$= \sqrt{0+1}$

$= 1$

~~Now~~ Now used the formula.

$$\boxed{\int_C f\,ds = \int_a^b f(\alpha(t))\,\|\alpha'(t)\|\,dt.}$$

$I_1 = \int_0^1 F(\alpha_1(t)) \cdot \|\alpha_1'(t)\|\,dt$

$= \int_0^1 (t+0) \cdot 1\,dt$

$= \int_0^1 t\,dt = \left.\frac{t^2}{2}\right]_0^1$

$= \frac{1}{2}.$

Similarly, $I_2 = \int_0^1 F(\alpha_2(t)) \cdot \|\alpha_2'(t)\|\, dt$.

$$= \int_0^1 (1 - t + t) \cdot \sqrt{2}\, dt$$

$$= \sqrt{2}$$

$$I_3 = \int_0^1 F(\alpha_3(t)) \, \|\alpha_3'(t)\|\, dt$$

$$= \int_0^1 (0 + 1 - t)\, 1\, dt$$

$$= \int_0^1 (1 - t)\, dt$$

$$= \frac{1}{2}.$$

i.e. $\int_C f\, ds = I_1 + I_2 + I_3$

$$= \frac{1}{2} + \sqrt{2} - \frac{1}{2}$$

$$\boxed{\int_C f\, ds = \sqrt{2}}$$

Fundamental theorem of Calculus :-

(1.) Let φ be a real function that is continuous on a closed interval $[a,b]$ and assume that, the integral $\int_a^b \varphi'(t)\,dt$ exist.

If φ' is continuous on the open interval (a,b), we have

$$\boxed{\int_a^b \varphi'(t)\,dt = \varphi(b) - \varphi(a)}$$

(2) Let φ be a differentiable scalars field with a continuous gradient $\nabla\varphi$ on an open connected set S in $\mathbb{R}^n$. Then for any two points a and b joined by a piecewise smooth path α in S, we have

$$\boxed{\int_a^b \nabla\varphi \cdot d\alpha = \varphi(b) - \varphi(a)}$$

When α is piecewise smooth, we partition the interval $[a,b]$ into a finite number (say r) of subinterval $[t_{k-1}, t_k]$, in each of which α is smooth

$$\int_a^b \nabla \varphi = \sum_{k=1}^{r} \int_{\alpha(t_{k-1})}^{\alpha(t_k)} \nabla \varphi$$

$$= \sum_{k=1}^{r} \{ \varphi [\alpha(t_k)] - \alpha [\alpha(t_{k-1})] \}$$

$$= \varphi(b) - \varphi(a)$$ as required.

Theorem :- Given $f(x,y) = P(x,y)\,i + Q(x,y)\,j$ where the partial derivatives $\frac{\partial P}{\partial y}$ and $\frac{\partial \varphi}{\partial x}$ are continuous on an open set S. If f is the gradient of some potential φ, then

$$\boxed{\frac{dP}{dy} = \frac{\partial Q}{\partial x}}$$ at each points of S.

Sol: Given $f(x,y) = (P(x,y), Q(x,y))$ — (1)

$$f = \nabla \phi$$

$$f(x,y) = \left(\frac{\partial \phi}{\partial x}, \frac{d\phi}{\partial y} \right) \quad (2)$$

Now comparing (1) and (2), we have

$$p(x,y) = \frac{\partial \phi}{\partial x} \quad \text{and} \quad Q(x,y) = \frac{\partial \phi}{\partial y}.$$

Taking partial derivative of P and Q with respect to y and x respectively, we have.

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right) = \frac{\partial^2 \phi}{\partial y \partial x}.$$

$$\frac{\partial P}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y}\right) = \frac{\partial^2 \phi}{\partial x \partial y}.$$

Now used the ~~claton~~ clairauts theorem :-

Suppose f is a real value function of two variables x, y and f(x,y) is defined on an open subsets U of $\mathbb{R}^2$. suppose ~~first~~ further that both the second-order mixed partial derivatives $f_{xy}(x,y)$ and $f_{yx}(x,y)$ exist and are continuous on U. Then

we have $f_{xy} = f_{yx}$.

$$\boxed{\frac{\partial f}{\partial x \partial y} = \frac{\partial f}{\partial y \partial x}}$$

From ~~clairaut~~ ~~clairauts~~. Clairauts theorem.

we have $\frac{\partial P}{\partial y} = \frac{\partial^2 \phi}{\partial y \partial x} = \frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial Q}{\partial x}$

at each points of S. i,e

$$\boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}}$$ ~~at~~ at each points of S.

Example :- $f(x,y) = y\hat{i} - x\hat{j}$ where $P = y$,

$Q = -x$.

$\frac{dP}{dy} = \frac{\partial y}{dy} = 1$

$\frac{dQ}{\partial x} = -1$

$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$.

i,e f is not gradient.

Q. find the work done by force

$F(x,y) = (8y^2+2)\hat{i} + 16x\hat{j}$. in ~~moving~~

moving a particles form $(-1,0)$ to $(1,0)$

along the upper half of the ellipse

$b^2x^2 + y^2 = b^2$. which values of b makes the

work a minimum.

Soln: $F(x,y) = (8y^2+2,\ 16x)$

$\gamma(t) = (\cos t, b\sin t)$

$b^2x^2 + y^2 = b^2$ — (1)

~~div~~ dividing (1) by b^2

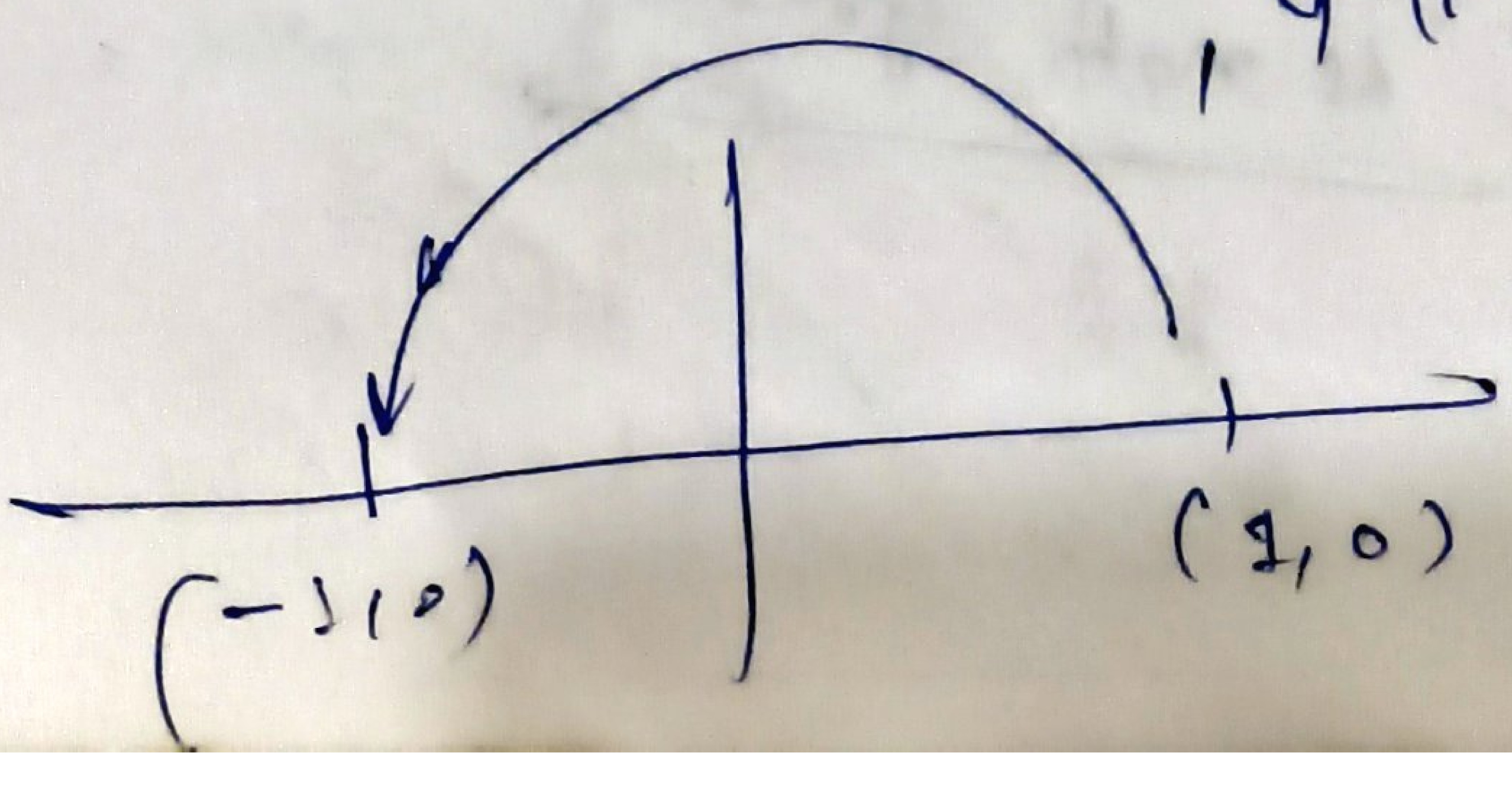
$$\frac{b^2x^2}{b^2} + \frac{y^2}{b^2} = \frac{b^2}{b^2}$$

$$\boxed{\frac{x^2}{1} + \frac{y^2}{b^2} = 1}$$

i,e $a=1$.

so $\gamma(t) = (\cos t, b\sin t)\ \forall\ t \in [0,\pi]$

, $\gamma'(t) = (-\sin t, b\cos t)$



So the work done is

$$W = \int_0^{\pi} F(\gamma(t)) \cdot \gamma'(t)\, dt$$

$$= \int_0^{\pi} (3\sin^2 t + 2,\ 16b\cos t) \cdot (-\sin t, b\cos t)\, dt.$$

$$= \int_0^{\pi} (-3\sin^3 t - 2\sin t + 16b^2\cos^2 t)\, dt.$$

$$= \int_0^{\pi} (-3\sin^3 t - 2\sin t + 16b^2(1 - \sin^2 t))\, dt.$$

used the formula $\sin 3t = 3\sin t - 4\sin^3 t$.

$$\boxed{4\sin^3 t = 3\sin t + \sin 3t.}$$

$$= 8b^2\pi - 8$$

→ If the work is minimum, then

$$\frac{dW}{db} = 0$$

$$\frac{d}{db}(8b^2\pi - 8) = 16b\pi = 0$$

$$\Rightarrow b = 0$$

Conservative vector fields :

Let $\vec{F} : D \longrightarrow \mathbb{R}^n$ be a vector field with domain $D \subseteq \mathbb{R}^n$. The vector field $\vec{F}$ is said to be conservative if is the gradient of a function.

or

There is a differentiable function $f : D \longrightarrow \mathbb{R}$ satisfying $\vec{F} = \nabla f$. Such a function f is called a potential function for $\vec{F}$.

Example : $\vec{F}(x,y) = (2x e^{xy} + x^2 y e^{xy})\hat{i} + (x^3 e^{xy} + 2y)\hat{j}$

Here $P = 2x e^{xy} + x^2 y e^{xy}$

$Q = x^3 e^{xy} + 2y.$

$$\frac{\partial P}{\partial y} = 3x^2 e^{xy} + x^3 y e^{xy}.$$

$$\frac{\partial Q}{\partial x} = 3x^2 e^{xy} + x^3 y e^{xy}$$

$$\boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}} \Rightarrow \text{So it conservative vector field.}$$

Necessary and sufficient condition for a vectors field to be gradient.

(1) $\vec{F}$ is conservative i,e f is the gradient of some potential function in S

(2) The line integral of f is independent of the path in S i,e $\int_C \vec{F} \cdot d\vec{r}$ is path ~~independe~~ independent mean that it only depend on the end points of the curve (C).

(3) The line integral of f is 0 around every piecewise smooth closed path in S i,e

$\oint_C \vec{F} \cdot d\vec{r} = 0$ around any closed curve C.

Q. Let S be the set of all $(x,y) \neq (0,0)$ in $\mathbb{R}^2$ and let f be the vector field defined on S by the equation

$$f(x,y) = \frac{-y}{x^2+y^2}\, i + \frac{x}{x^2+y^2}\, j.$$

Is f is gradient on S?

Soln: f is not gradient on S.

We will now compute the line integral of f around the unit circle given by

$$\alpha(t) = \cos t\, i + \sin t\, j, \quad 0 \leq t \leq 2\pi.$$

$$\oint_C f \cdot d\alpha = \int_0^{2\pi} f(\alpha) \cdot \alpha'(t)\, dt.$$

$$= \int_0^{2\pi} (\sin^2 t + \cos^2 t)\, dt$$

Because $f(x,y) = f(\cos t, \sin t)$

$$= \left(\frac{-\sin t}{\cos^2 t + \sin^2 t}, \frac{\cos t}{\sin^2 t + \cos^2 t} \right)$$

$$\alpha'(t) = (-\sin t, \cos t)$$

$$\oint f \cdot d\alpha = 2\pi = \int_0^{2\pi} 1 \cdot dt.$$

But the line integral around the closed path is not zero., so f is not gradient on S.

i/e $\oint_C f \cdot d\alpha \neq 0 \Rightarrow f$ is not gradient.

Construction of Potential Functions

Example: find a potential function ϕ for the vector fields defined on $\mathbb{R}^3$ by the equation.

$f(x,y) = xi + yj$

Soln: $\cancel{\frac{\partial \phi}{\partial x} = x}$, $\cancel{\frac{\partial \phi}{\partial y} = y}$.

$f(x,y) = xi + yj$

$f(x,y) = P(x,y)\hat{i} + Q(x,y)\hat{j}$

$\Rightarrow P = x$ and $Q = y$.

For gradient, we must have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

so $\frac{\partial P}{\partial y} = \frac{\partial x}{\partial y} = 0$ and $\frac{\partial Q}{\partial x} = \frac{\partial y}{\partial x} = 0$

Since $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ implies f is gradient.

$\Rightarrow$ ~~implies~~

To check the potential we have

$$\phi_1(x,y) = \int p(x,y)\,dx + A(y)$$

$$= \int x\,dx + A(y)$$

$$= \frac{x^2}{2} + A(y)$$

$$\phi_2(x,y) = \cancel{\phi f(x,y)} \int Q(x,y)\,dy + B(x)$$

$$= \int y\,dy + B(x)$$

$$= \frac{y^2}{2} + B(x)$$

By inspection, we have

$$A(y) = \frac{y^2}{2} \text{ and } B(x) = \frac{x^2}{2}$$

$$\phi(x,y) = \phi_1(x,y) + \phi_2(x,y)$$

$$= \frac{x^4}{2} + \frac{y^2}{2} + \frac{x^2}{2} + c.$$

Q Find a potential function φ for the vector field defined on $\mathbb{R}^3$ by the equation

$$f(x,y,z) = (2xyz + z^2 - 2y^2 + 1)\,i + (x^2 z - 4xy)\,j + (x^2 y + 2xz - 2)\,\hat{k}$$

Soln.

$$f(x,y,z) = P(x,y,z)\,\hat{i} + Q(x,y,z)\,\hat{j} + R(x,y,z)\,\hat{k}.$$

$$P = 2xyz + z^2 - 2y^2 + 1$$

$$Q = \cancel{x^3 z - 4xy} \quad x^2 z - 4xy.$$

$$R = (x^2 y + 2xz - 2)$$

$$\frac{dP}{dy} = 2xz - 4y.$$

$$\frac{\partial Q}{\partial x} = \cancel{3x^2 z - 4y}, \quad 2xz - 4y.$$

similarly, $\dfrac{\partial Q}{\partial z} = \dfrac{\partial R}{\partial y}$, and $\dfrac{\partial P}{\partial z} = \dfrac{\partial R}{\partial x}$.

Here f is gradient of some scalar field for potential φ.

$$\phi_1(x,y,z) = \int P(x,y,z)\,dx + A(y,z)$$

$$= \int (2xyz + z^2 - 2y^2 + 1)\,dx + A(y,z)$$

$$= x^2yz + z^2x - 2y^2x + x + A(y,z)$$

$$\phi_2(x,y,z) = \int Q(x,y,z)\,dy + B(x,z)$$

$$= \int (x^2z - 4xy)\,dy + B(x,z)$$

$$= x^2yz - 2xy^2 + B(x,z)$$

$$\phi_3(x,y,z) = \int R(x,y,z)\,dz + C(x,y)$$

$$= \int (x^2y + 2xz - 2)\,dz + C(x,y)$$

$$= x^2yz + xz^2 - 2z + c(x,y)$$

Using the method inspection

$A(y,z) =$ Common variables in $\phi_2(x,y,z)$ and $\phi_3(x,y,z)$

$\cancel{\phi_2(y)}$

ϕ_2 $\qquad$ (ϕ_3)

$= \left(x^2yz - 2xy^2\right)$ $\qquad$ $\left(x^2yz + xz^2 - 2z\right)$

Contain x $\qquad$ Contain x

$A(y,z)$ doesnot contain x, so we have

$A(y,z) = -2z$

Similarly $B(x,z) = \phi_1(x,y,z)$ $\qquad$ $\phi_3(x,y,z)$

$x^2zy + z^2x - 2y^2x + x$ $\qquad$ $x^2yz + xz^2 - 2z.$

containg $\qquad$ Contain (x,z) $\qquad$ Containg. $\qquad$ Contain $x,z.$

Inspection Method

In $B(x,z)$ = Those variable contain y are not eligible

so $B(x,z) = xz^2 + x - 2z.$

similarly $C(x,y) = \underbrace{\phi_1(x,y,z) \text{ and } \phi_2(x,y,z)}_{\text{common variable.}}$

$C(x,y) = x^2yz + xz^2 - 2xy^2 + x \rightarrow \phi_1$

Contain Z

$x^2yz \cancel{+z} - 2xy^2 \rightarrow \phi_2$

Contain Z.

$C(x,y)$ = those variable contain Z are not eligible.

so $c(x,y) = x - 2xy^2$

i.e $A(y,z) = -2z$

$B(x,z) = xz^2 + x - 2z.$

$c(x,y) = x - 2xy^2$

so Required ~~pot~~ potential function on $\mathbb{R}^3$.

$\phi(x,y,z) = x^2yz + xz^2 - 2xy^2 + x - 2z.$

is a potential for f on $\mathbb{R}^3$.

Q. find a potential function ϕ for the vector field defined on $\mathbb{R}^3$ by the equation

$$f(x,y,z) = (x+z)i - (y+z)j + (x-y)k$$

Ans:- $\phi_1(x,y,z) = \int p(x,y,z)\,dx + A(y)$

$$= \int (x+z)\,dx + A(z,y)$$

$$= \frac{x^2}{2} + xz + A(z,y)$$

Similarly $\phi_2(x,y,z) = -\frac{y^2}{2} + yz + B(x,z)$

$$\phi_3(x,y,z) = xz - yz + C(x,y)$$

By using the method inspection.

$$A(z,y) = -\frac{y^2}{2} - yz$$

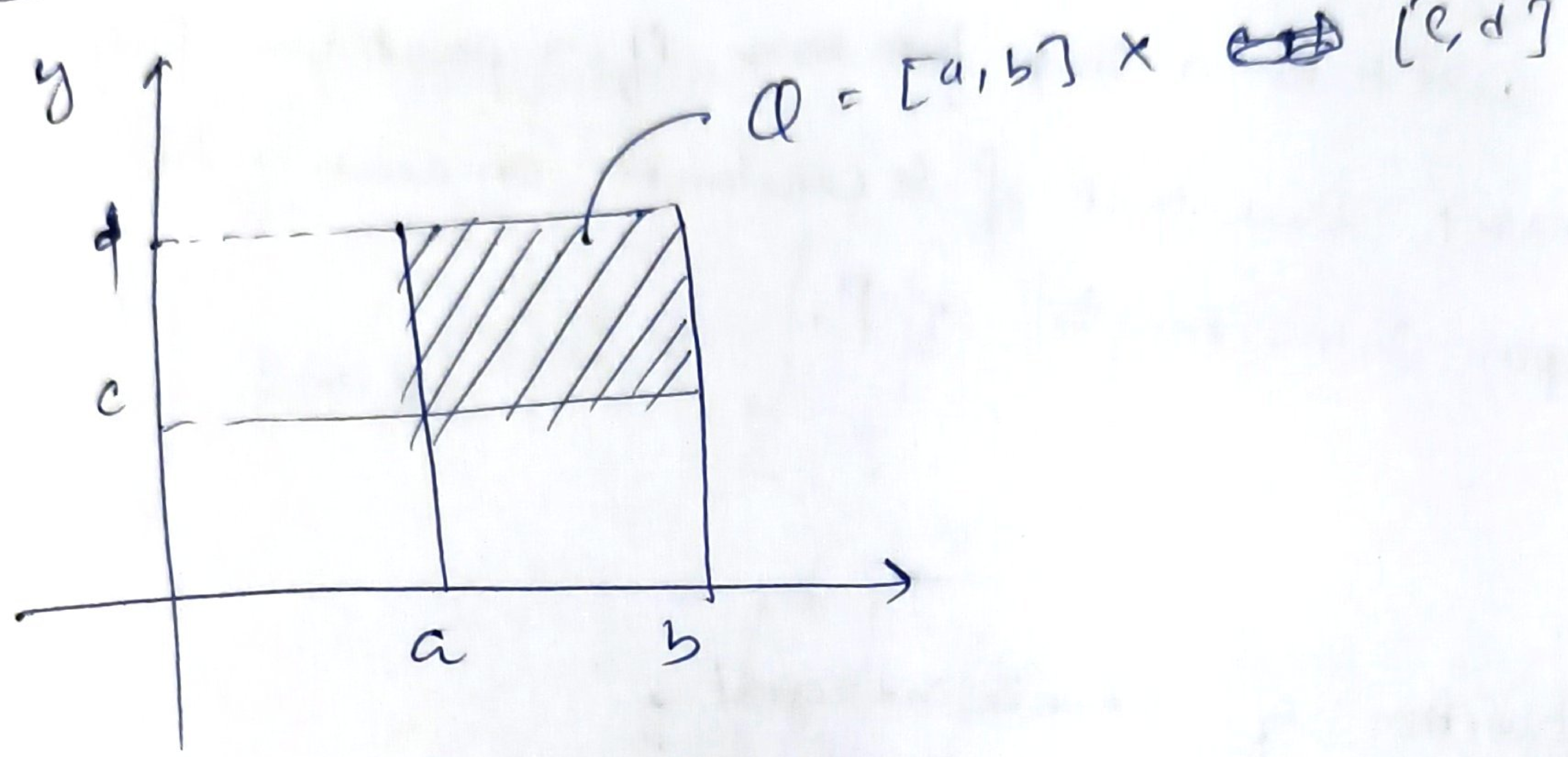
$$B(x,z) = \frac{x^2}{2} + xz$$

$$c(x,y) = \left(\frac{x^2-y^2}{2}\right).$$

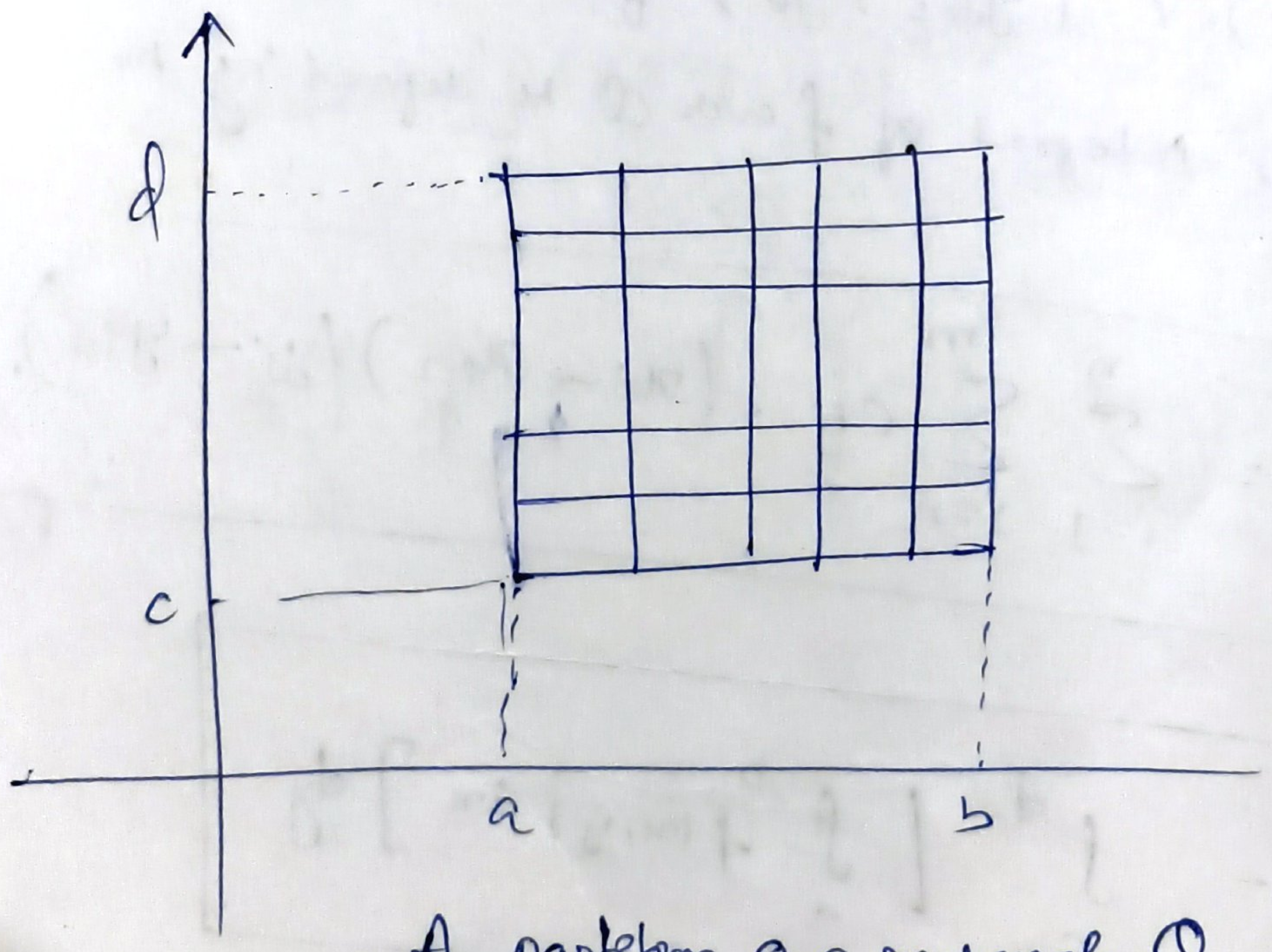
$$\therefore \phi(x,y,z) = \frac{x^2}{2} - \frac{y^2}{2} - yz + xz + c$$

is the required potential fn on $\mathbb{R}^3$.

Step Function :-



A rectangle Q, the cartesian product of two interval.



A partition of a rectangle Q.

A function f defined on a rectangle $Q = [a,b] \times [c,d]$ is said to be a step function if a partition P of Q exist such that f is constant on each of the open subrectangles of P.

Step function of Double integral :-

Let f be a step function which takes the constant c_{ij} on the open subrectangle $(x_{i-1}, x_i) \times (y_{j-1}, y_j)$ of a rectangle Q.

$\therefore$ The double integral of f over Q is defined by the formula

$$\boxed{\iint_Q f = \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} \cdot (x_i - x_{i-1})(y_j - y_{j-1}).}$$

$$\boxed{\iint_Q f = \int_c^d \left[\int_a^b f(x,y)\, dx \right] dy}$$

Iterated integral :- / ~~Fubini~~ Fubini's theorem.

If $f(x,y)$ is continuous on $R = [a,b] \times [c,d]$, then

$$\iint_R f(x,y)\, dA = \int_a^b \int_c^d f(x,y)\, dy\, dx$$

$$= \int_c^d \int_a^b f(x,y)\, dx\, dy.$$

Q. If $Q = [-1,1] \times [0, \pi/2]$.

Evaluate $\iint_Q (x\sin y - ye^x)\, dx\, dy$

$\Rightarrow$ integrate first with respect to y.

Soln:

$$\iint_Q f = \int_{-1}^{1} \left[\int_0^{\pi/2} (x\sin y - ye^x)\, dy \right] dx.$$

$$= \int_{-1}^{1} \left(-x\cos y - \frac{y^2}{2} \cdot e^x \right]_0^{\pi/2} dx.$$

$$= \int_{-1}^{1} \left(-\pi^2 \cdot \frac{e^x}{8} + x \right) dx = \left(\frac{1}{e} - e \right) \frac{\pi^2}{8}$$

~~integer integrate first with respect to y~~

we can also integrate first with respect to x.

$$\iint_Q (x\sin x - ye^x)\,dx\,dy = \int_0^{\pi/2}\left(\int_{-1}^{1}(x\sin y - ye^x)\,dx\right)dy$$

$$= \int_0^{\pi/2}\left(-ey + \frac{y}{e}\right)dy.$$

$$= \left(1/e - e\right)\int_0^{\pi/2}\left(-ey + \frac{y}{e}\right)dy. = \int_0^{\pi/2} y\,dy \cdot \left(\frac{1}{e} - e\right)$$

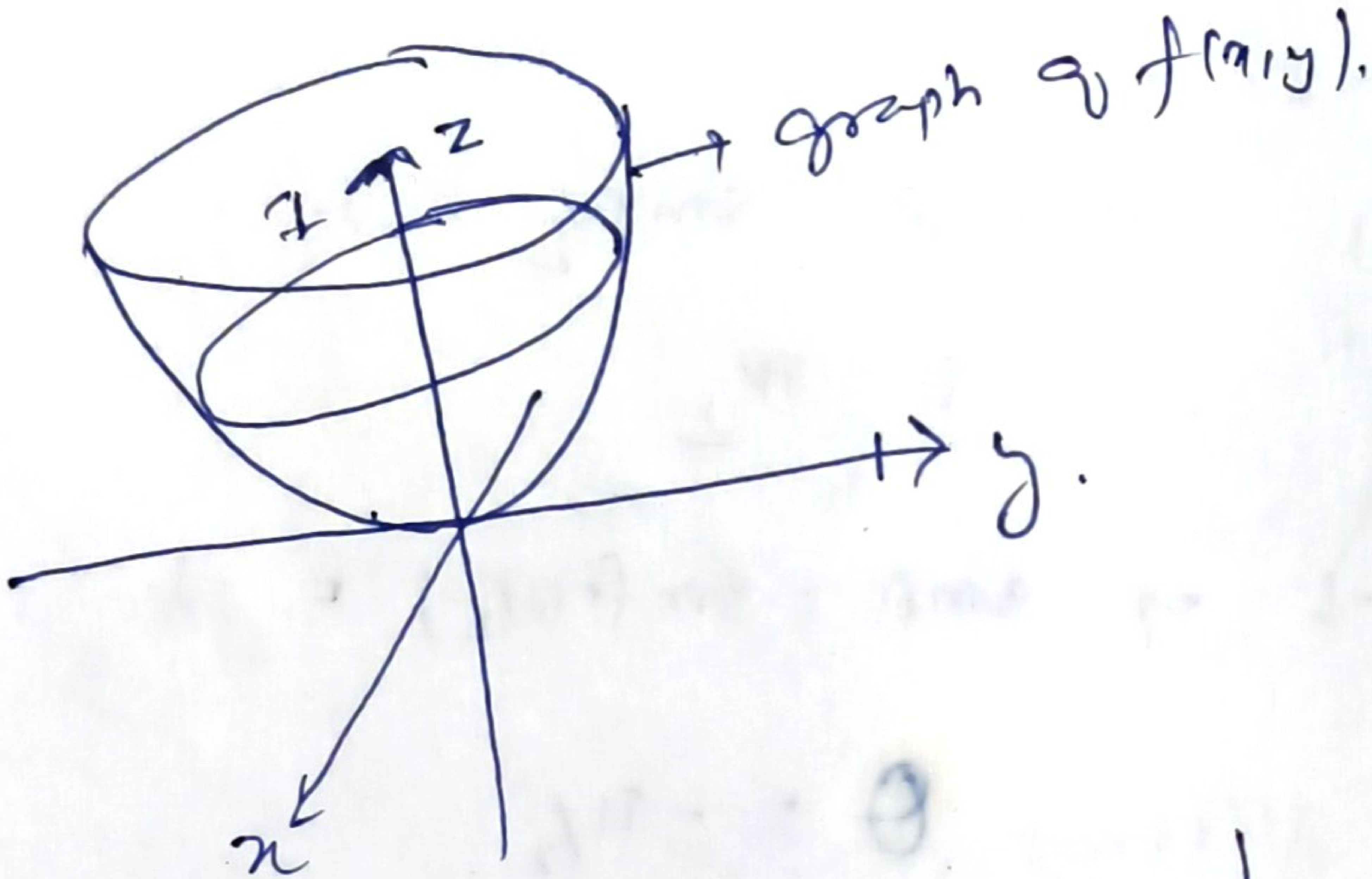
$$= \left(\frac{1}{e} - e\right)\frac{\pi^2}{8}$$

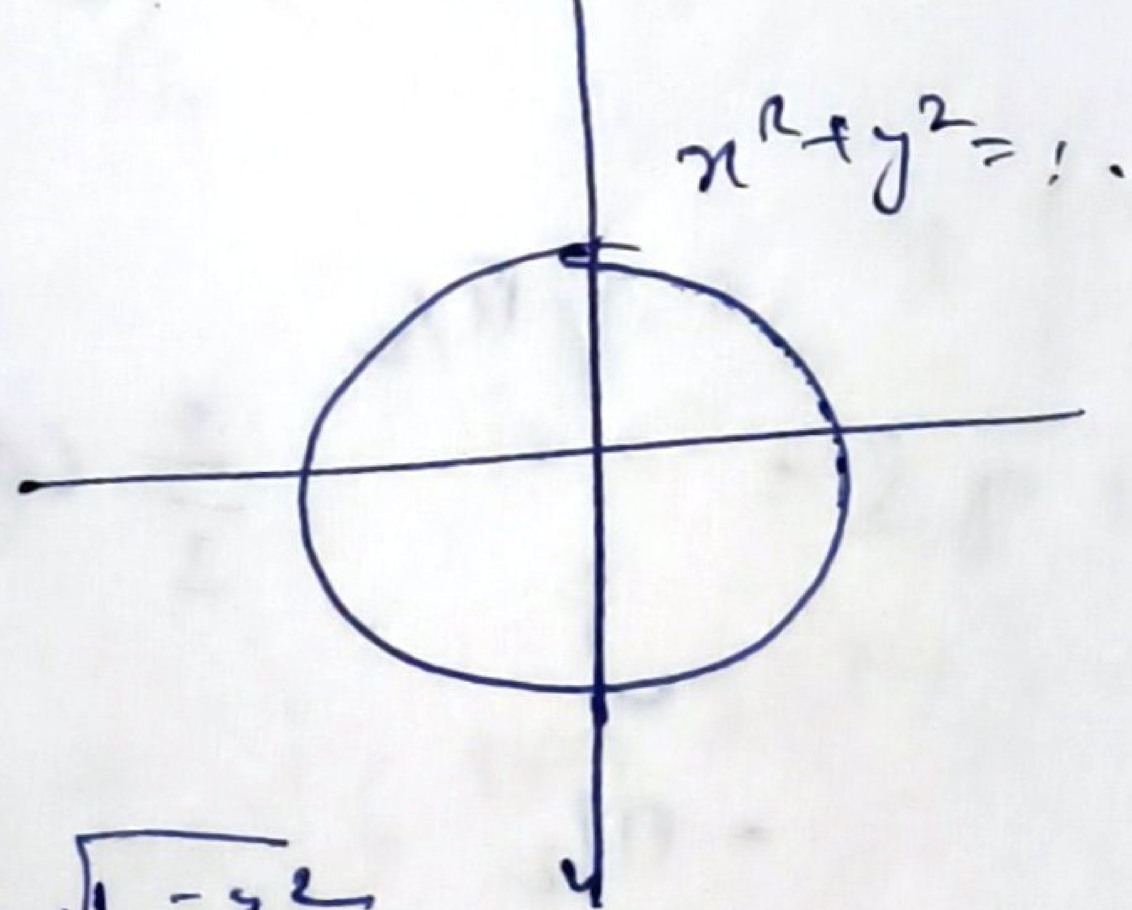
Q let f defined on the rectangle

$$Q = [-1,1] \times [-1,1] \quad \text{and}$$

$$f(x,y) = \begin{cases} x^2+y^2 & \text{if } x^2+y^2 \le 1. \\ 0 & \text{otherwise.} \end{cases}$$

Soln:





$$\text{volume of } S = \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} (x^2+y^2)\, dx\, dy$$

$$= \int_{-1}^{1} \left[\frac{x^3}{3} + xy^2 \right]_{-\sqrt{1-y^2}}^{+\sqrt{1-y^2}} dy.$$

$$= \int_{-1}^{1} \frac{2}{3}(1-y^2)^{3/2} + 2y^2(1-y^2)^{1/2}\,dy.$$

put $y = \sin\theta$

$dy = \cos\theta \, d\theta.$

$\Rightarrow \; y = 1 \;\Rightarrow\; \sin\theta = \sin \pi/2 = 1$

$\theta = \pi/2$

$y = -1 \;\Rightarrow\; \sin\theta = \sin(-\pi/2) = -1.$

$\theta = -\pi/2$

$$\text{Volume of } S = \int_{-\pi/2}^{\pi/2} \frac{2}{3}\cos^4\theta + 2\sin^2\theta\cos^2\theta \, d\theta$$

$$= \int_{-\pi/2}^{\pi} \frac{1}{6}(1+\cos 2\theta)^2 + \frac{1}{2}\sin^2 2\theta)\, d\theta.$$

Note:

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$\cos^4\theta = \frac{1+\cos 2\theta}{2}$$

$\sin 2\theta$
$= 2\sin\theta\cos\theta$

$$= \int_{-\pi/2}^{\pi/2} \frac{1}{6} + \frac{1}{6}\cos^2 2\theta + \frac{1}{3}\cos 2\theta + \frac{1}{2}\sin^2 2\theta \, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \frac{1}{6} + \frac{1}{12}(1+\cos 4\theta) + \frac{1}{3}\cos 2\theta + \frac{1}{4}(1-\cos 4\theta)\, d\theta.$$

$$= \left[\frac{1}{6}\theta + \frac{1}{12}\left(\theta + \frac{\sin 4\theta}{4}\right) + \frac{1}{6}\sin 2\theta + \frac{1}{4}(1-\cos 4\theta)\right]_{-\pi/2}^{\pi/2}$$

$$= \frac{1}{6}\pi/2 + \frac{1}{12}\left(\pi/2 + \frac{\sin 4\times\pi/2}{4}\right) + \frac{1}{6}\sin 2\times\pi/2 + \frac{1}{4}(1-\cos 4\times\pi/2)$$

$$- \left[\frac{1}{6}(-\pi/2) + \left(\frac{1}{12}\left(-\pi/2 + \frac{\sin(4(-\pi/2))}{4}\right) + \frac{1}{6}\sin 2(-\pi/4) + \frac{1}{4}\right.\right.$$

$$\frac{1}{4}\left(1-\cos 4(-\pi/2)\right)\Big)$$

$$= \pi/2$$

Q. Let f defined on the rectangle $Q = [0,1] \times [0,1]$ as follow

$$f(x,y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y. \end{cases}$$

find $\iint_Q f(x,y)\,dx\,dy$.

Soln.

$$\int_0^1 \int_0^1 f(x,y)\,dx\,dy.$$

$$\int_0^1 \left(\int_0^1 f_x(y)\,dy \right) dx.$$

$$f(x,y) = I_{\{x\}}(y), \quad \text{where } x \in [0,1]$$

where $I_{\{x\}}$ denotes the indicator function.

~~$f(x,y)$~~ i/e $I_{\{x\}}(y) = \begin{cases} 1 & \text{if } y \in \{x\} \\ 0 & \text{if } y \notin \{x\}. \end{cases}$

$$= \int_0^1 0 \cdot dx \quad \text{since } \boxed{x \neq y}$$

$$= 0$$

$$\Rightarrow \int_0^1 f_x(y)\,dy = \int_0^1\int_0^1 f(x,y)\,dx\,dy.$$

$$\Rightarrow \int_0^x f_x(y)\,dy + \int_x^1 f_x(y)\,dy = \int_0^1\int_0^1 f(x,y)\,dx\,dy.$$

$$\Rightarrow 0 + 0 = \int_0^1\int_0^1 f(x,y)\,dx\,dy.$$

$$\Rightarrow \int_0^1\int_0^1 f(x,y)\,dx\,dy = 0$$

Integrability of bounded function with discontinuities

Bounded set of Content zero:

Let A be a bounded subset of the plane. The set A is said to have content zero if for every $\varepsilon > 0$ there is a finite set of rectangles whose union contain A and the sum of whose areas doesnot exceed ε.

Note:- A rectangle is any set of the form $R = [a,b] \times [c,d]$

Area of R (rectangle) $= (b-a)(d-c)$

i,e a bounded set $A \subset \mathbb{R}^2$ has zero content if for any $\varepsilon > 0$ one can find a finite number of rectangle $R_1, R_2, \ldots R_m \subset \mathbb{R}^2$ such that $A \subset R_1 \cup \ldots \cup R_m$.

and $\text{area}(R_1) + \text{area}(R_2) + \ldots + \text{area}(R_m) < \varepsilon$

Q. Compute the volume of the solid enclosed by the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

soln: We know that solid lies in between ~~x and y axes~~ so it lies between the graph of two functions f and g, where

$$g(x,y) = c\sqrt{1 - \frac{x^2}{a^2} - y^2/b^2}$$

Since $\frac{z^2}{c^2} = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}$

$$\boxed{z = c\sqrt{1 - x^2/a^2 - y^2/b^2}}$$

and $f(x,y) = -g(x,y)$ because

$(x,y,z) \in E$ ~~and~~ Here E is ellipsoid.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{c^2}{z^2} \leq 1$$

$\Rightarrow \quad z^2 \le g(x,y)^2$

$\Rightarrow \quad |z| \le g(x,y)$

$\Rightarrow \quad -g(x,y) \le z \le g(x,y)$

Therefore $f(x,y) \le z \le g(x,y)$

where ~~$f(x,y) = -g(x,y z)$~~

$f(x,y) = -g(x,y)$

Now used the theorem.

$$\iint_S f(x,y)\,dx\,dy = \int_a^b \left[\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy\right] dx$$

$$\text{volume}, V = \iint_S (g-f)$$

$$= \iint_S g - (-g)$$

$$= 2 \iint_S g$$

$$= 2 \int_0^a \left[\int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\, dy \right] dx \quad \text{(1)}$$

~~State~~ Since $(x,y) \in S$, Here S is the elliptical region is given by

$$S = \left\{ (x,y) \;\middle|\; \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \right\}.$$

Let put $A = \sqrt{1 - \frac{x^2}{a^2}}$

from (1), Take the inner integral

$$\int_0^{bA} \sqrt{A^2 - \frac{y^2}{b^2}}\, dy = A \int_0^{bA} \sqrt{1 - \frac{y^2}{(Ab)^2}}\, dy$$

using the change of variable

$y = Ab\sin t$

$\frac{dy}{dt} = Ab\cos t$

$dy = Ab\cos t\,dt$

$y = bA = Ab\sin t$

$\sin t = 1$

$t = \pi/2$

$y = 0 = Ab\sin t$

$\sin t = 0$

$t = 0$

$= \cancel{A \int_0^{\pi/2} \sqrt{1 - \sin^2 t}\, dy}.$

$= A \int_0^{bA} \sqrt{1 - \frac{y^2}{(Ab)^2}}\, dy.$

$= \cancel{A \int_0^{bA} \sqrt{1 -}}.$

$= A \int_0^{\pi/2} \sqrt{1 - \frac{A^2 b^2 \sin^2 t}{(Ab)^2}} \cdot Ab\cos t\,dt$

$= A^2 b \int_0^{\pi/2} \sqrt{1 - \sin^2 t}\, \cos t\, dt.$

$$= A^2 b \int_0^{\pi/2} \cos^2 t \, dt = A^2 b \int_0^{\pi/2} (1 + \cos 2t) \, dt.$$

$$= A^2 b \cdot \frac{\pi}{4}. = \frac{\cancel{\pi} b}{4} \cdot \left(1 - \frac{x^2}{a^2}\right)$$

Since $\cos^2 t = 1 + \cos 2t$

Now again from (1),

$$= \cancel{2} 8c \int_0^a \left(\int_0^{b\sqrt{1 - \frac{x^2}{a^2}}} \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} \, dy \right) dx.$$

$$\cancel{\sigma} = \cancel{2} 8c \int_0^a \left(A^2 b \cdot \pi/4 \right) dx.$$

$$= \frac{\cancel{2}}{8c} \frac{b \cdot \pi}{4} \int_0^{\cancel{A} a} A^2 \, dx.$$

$$= 8c \cancel{\frac{\pi b}{4}} 8c \int_0^a \frac{\pi b}{4} \left(1 - \frac{x^2}{a^2}\right) dx$$

$$= \frac{\pi b c}{\cancel{2}} \; 8c \left(\cancel{\int_0^a \frac{\pi b}{4} dx} - \frac{x^3}{3a^2} \right|$$

$$= 8c\int_0^a \frac{\pi b}{4}\left(1 - \frac{x^2}{a^2}\right)dx$$

$$= 8c\frac{\pi b}{4}\int_0^a \left(1 - \frac{x^2}{a^2}\right)dx.$$

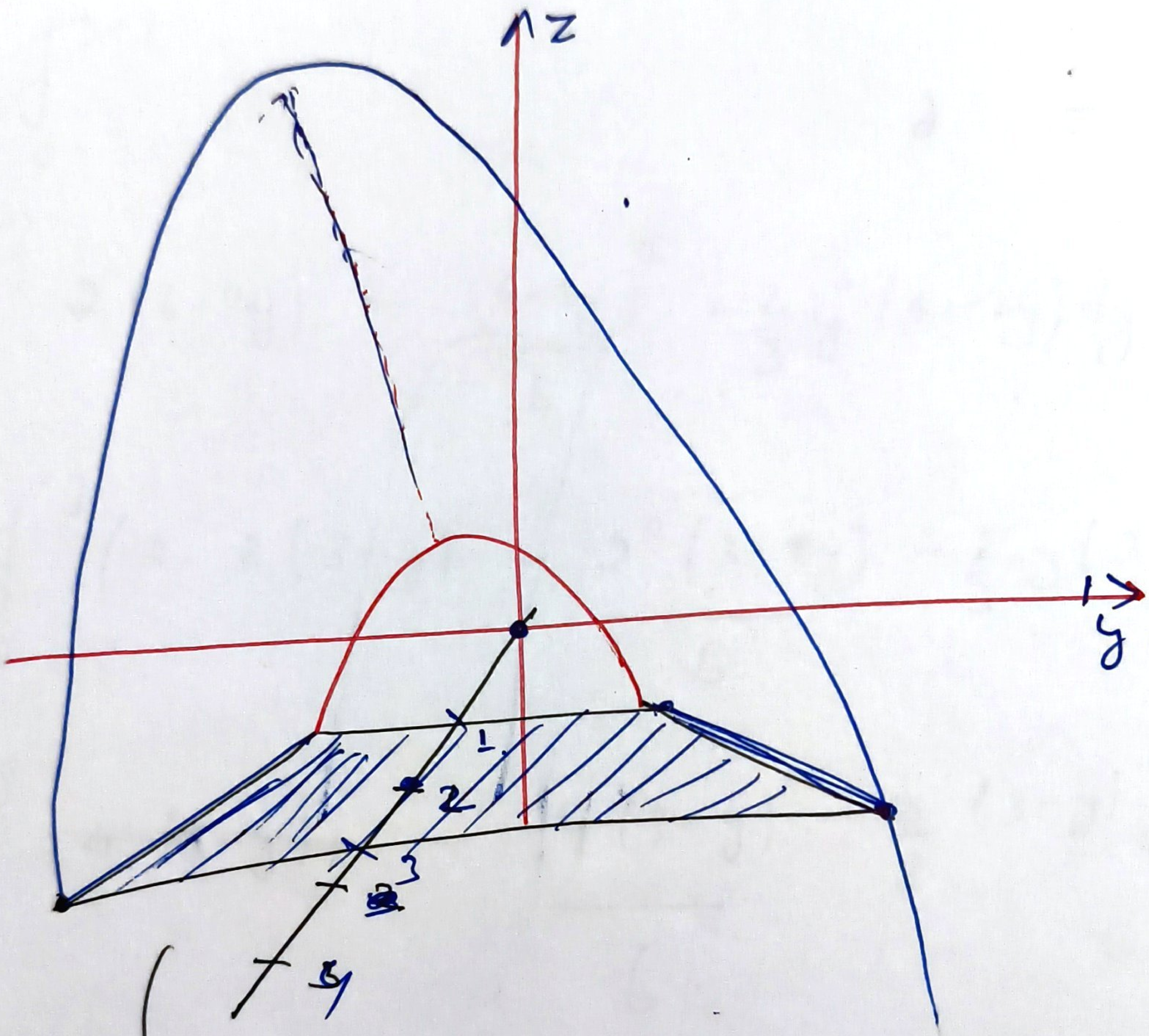
$$= 8c\frac{\pi b}{4}\left[(a-0) - \frac{a^3}{3a^2}\right].$$

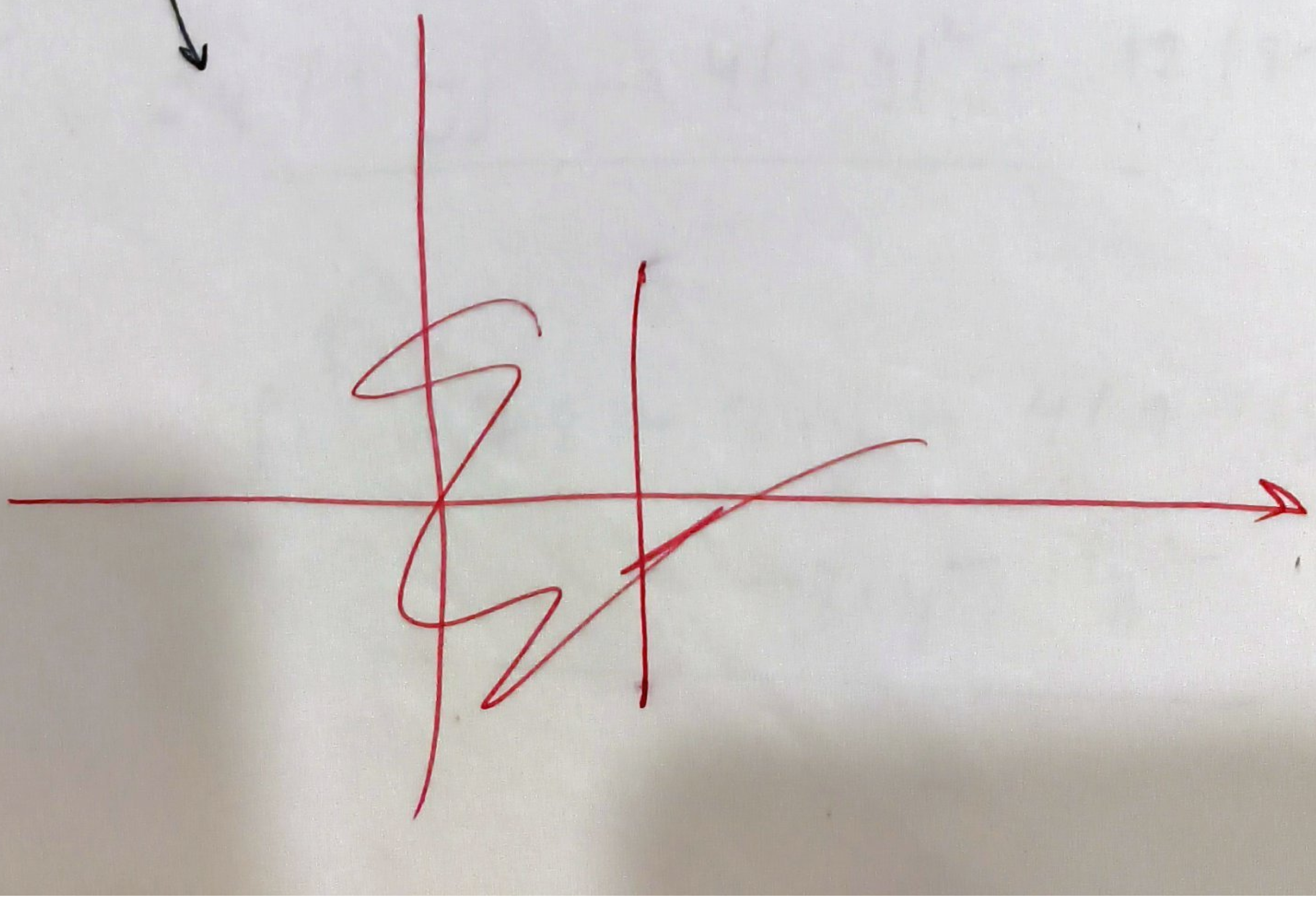
$$= \frac{\cancel{8}^{2}\,\pi bc}{\cancel{4}}\left[\frac{3a^3 - a^3}{3a^2}\right)$$

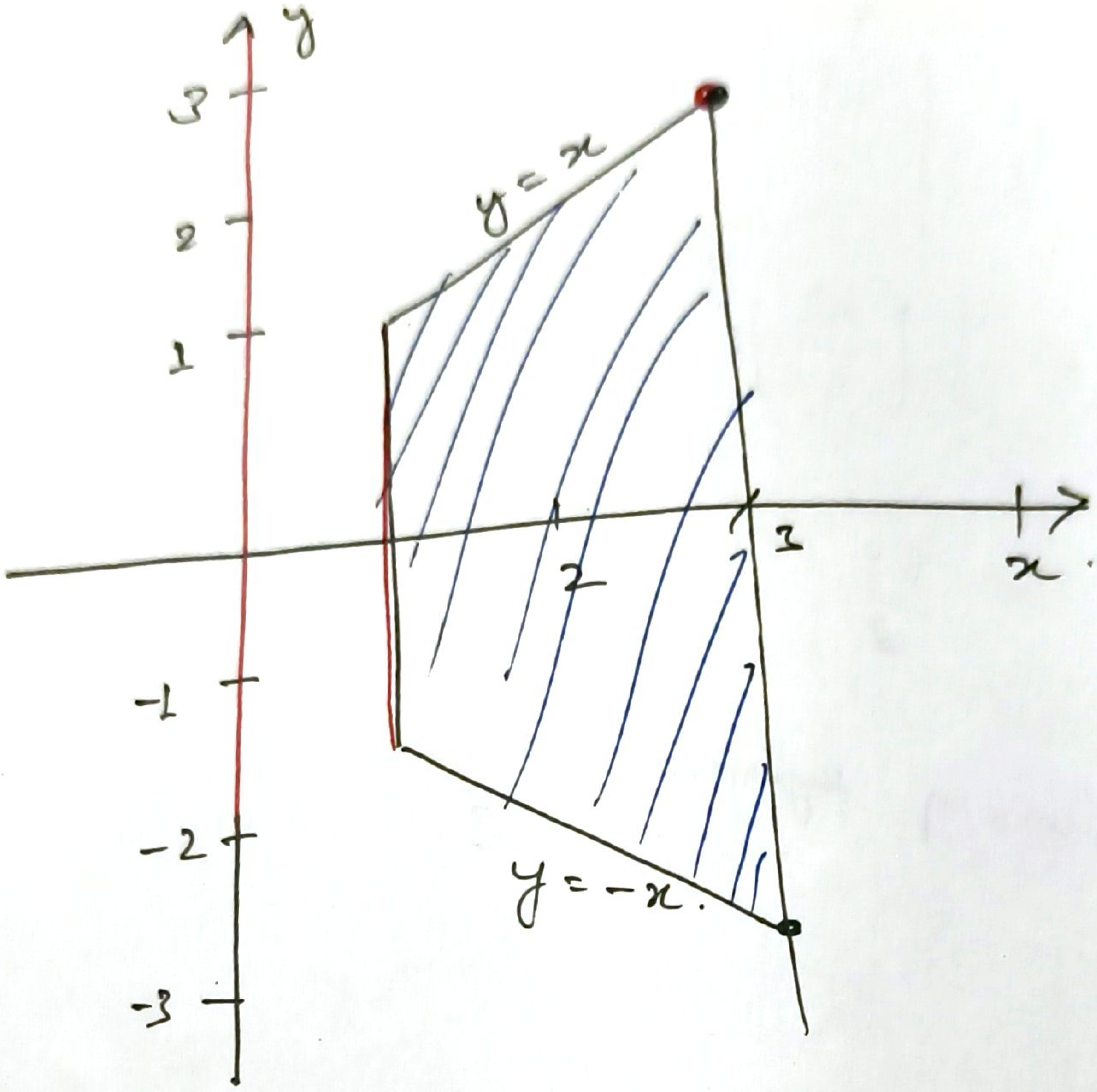
$$= \frac{4}{3}\pi abc.$$

Q. A Solid is bounded by the surface $z = x^2 - y^2$, the xy-plane, and the plane $x = 1$ and $x = 3$. Make a sketch of the solid and compute its volume by double integration.

Soln:







Therefore volume of pyramid $= \displaystyle\int_1^3 \int_{-x}^{x} \int_0^{x^2-y^2} dz\,.\,dy\,.\,dx$

$$= \int_1^3 \int_{-x}^{x} \int_0^{x^2-y^2} dz\,.\,dy\,.\,dx\,.$$

$$= \int_1^3 \int_{-x}^{x} (x^2-y^2)\,dy\,dx\,.$$

$$= \int_1^3 \left[x^2y - \frac{1}{3}y^3\right]_{-x}^{x} dx$$

$$= \int_1^3 \cancel{\left[x^3 - \frac{1}{3}x^3 + x^3 + \frac{1}{3}x^3\right]}$$

$$= \frac{4}{3}\int_1^3 x^3\,dx$$

$$= \frac{4}{3}\left[\frac{x^4}{4}\right]_1^3$$

$$= \frac{80}{3}.$$

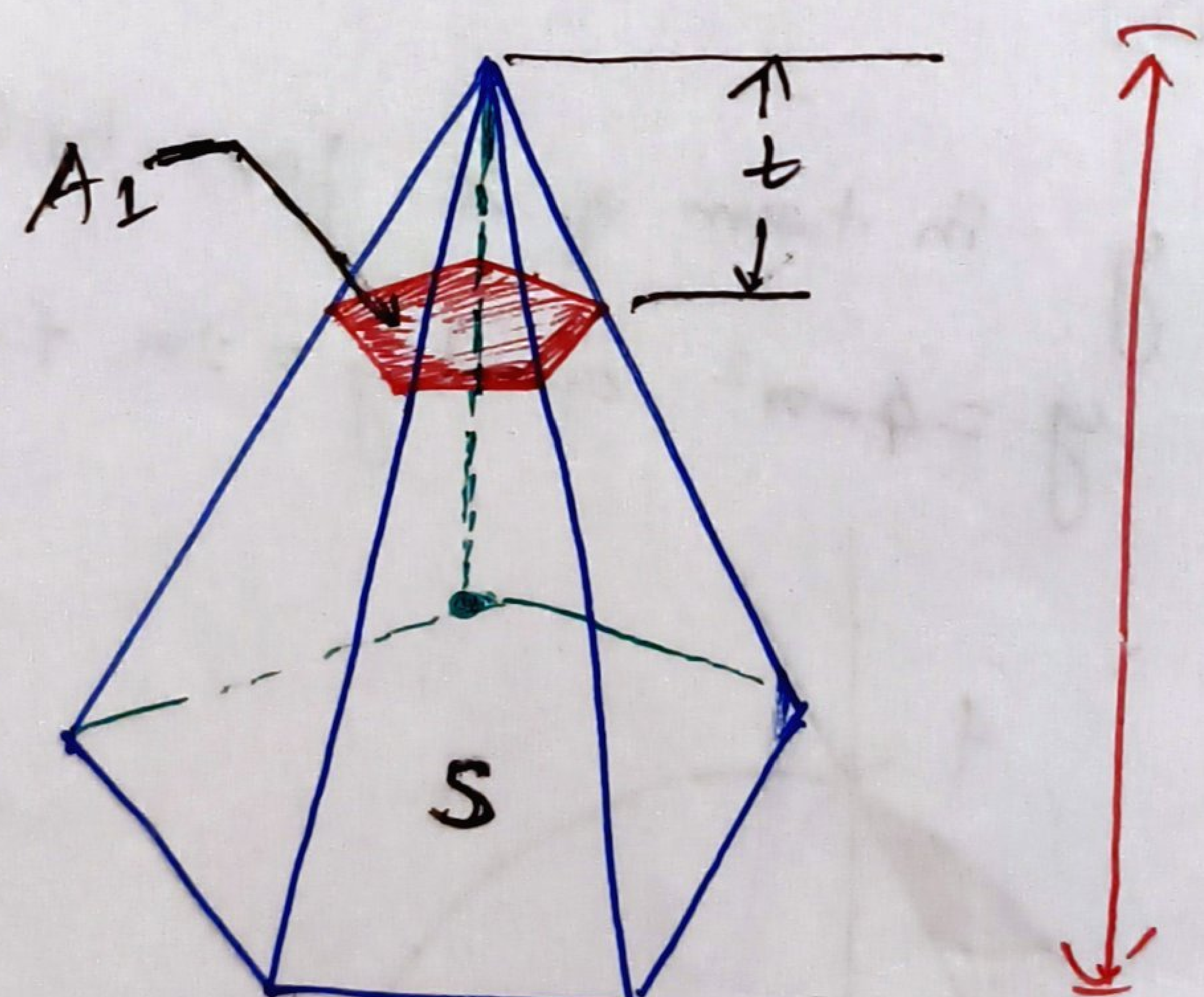
Volume of pyramid $= \frac{80}{3}$.

Q. A solid cone is obtained by connecting every point of a plane region S with a vertex not in the plane of S. Let A denote the area of S, and let h denote the altitude of the cone.

Prove that

(1) The cross-sectional area cut by a plane parallel to the base and at a distance from t from the vertex is $\left(\frac{t}{h}\right)^2 A$, if $0 \le t \le h$

Soln:



If a cutting plane parallel to the base will pass through the cone, the smaller cone will be formed and that ~~which~~ will be similar to the original cone.

Now using similar solids theorem :

If two similar solid have a scale factor of a:b, then corresponding areas have a ~~ratt~~ ratio of $a^2 : b^2$ and corresponding volume have a ratio of $a^3 : b^3$.

Note:- The terms areas in the theorem refer to any pair of corresponding areas in the similar solids, such as lateral areas, base areas, and surface areas.

$$\Rightarrow \frac{A_L}{A} = \frac{t^2}{h^2}$$

$$\Rightarrow A_1 = \frac{A t^2}{h^2}$$

$$\boxed{A_1 = \left(\frac{t}{h}\right)^2 A}$$