Topological Manifold :-

M is a topological of dimension n or a topological n-manifold if it has the following properties.

① M is a Hausdorff space. For every pair of distinct point $p, q \in M$, there are disjoint open subsets $U, V \subseteq M$ such that $p \in U$ and $q \in V$

② M is second-countable i,e there exist a countable basis for the topology of M.

③ M is locally Euclidean of dimension n i,e for every $p \in M$, there exist a triple $\{\varphi, U, V\}$ called a chart (around p) where U is an open neighborhood of p in M, V an open subsets of $\mathbb{R}^n$, and ~~ep~~ $\varphi : U \longrightarrow V$ a homeomorphism

i.e for each $p \in M$ we can find

→ an open subset $U \subseteq M$ containing p,

→ an open subset $\hat{U} \subseteq \mathbb{R}^n$, and

→ a homeomorphism $\varphi : U \longrightarrow \hat{U}$

Topological Invariance of Dimension :

~~An~~ A non-empty n-dimensional topological manifold cannot be homeomorphic to an m-dimensional manifold unless $m = n$.

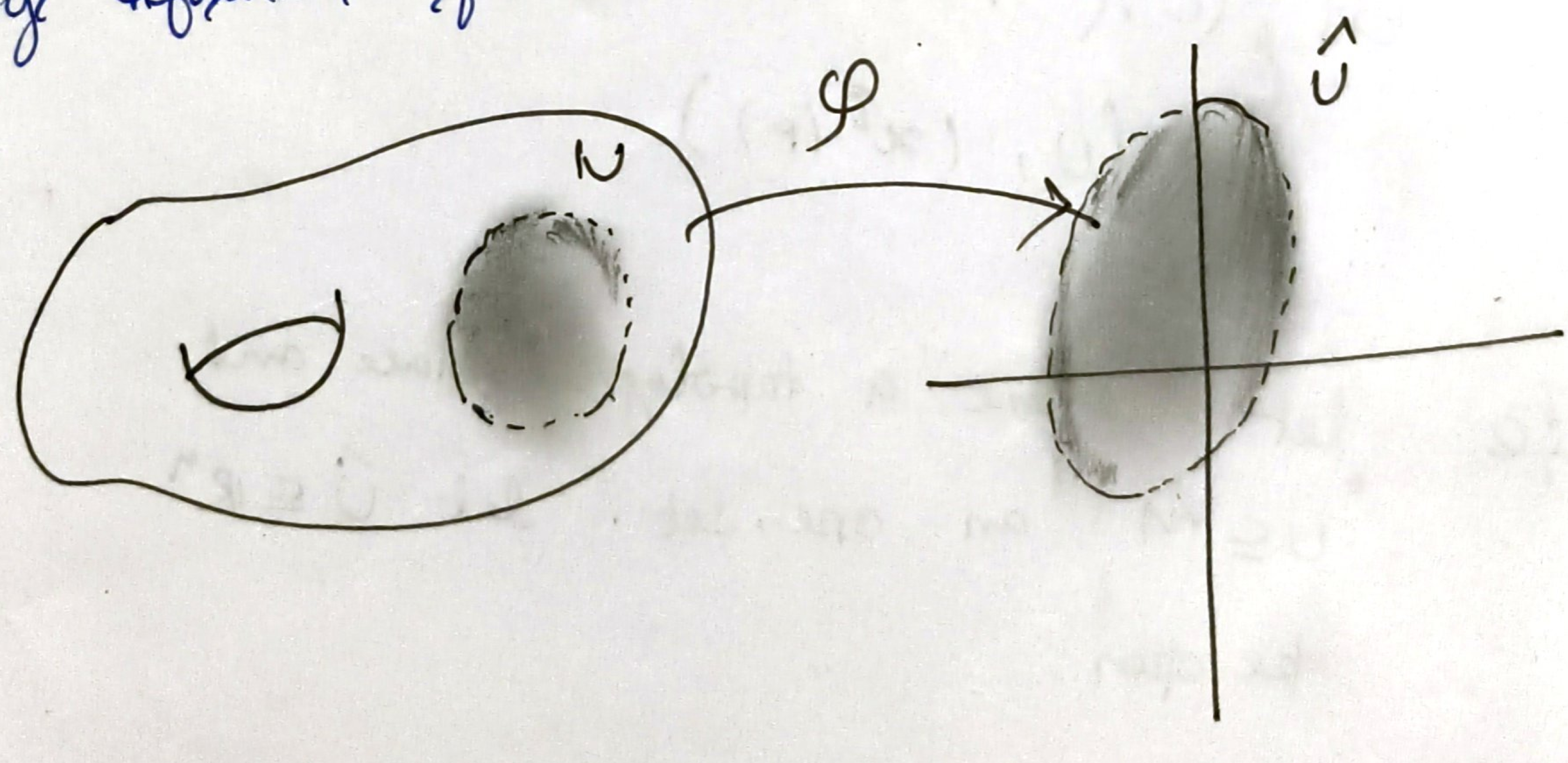
Coordinate charts :-

let M be a topological manifold or topological n-manifold.

A coordinate chart (or chart) on M is a pair (U, φ) where U is an open subset of M and $\varphi : U \longrightarrow \hat{U}$ is a homeomorphism from U to an open subset

$$\hat{U} = \varphi(U) \subseteq \mathbb{R}^n.$$

By definition of a topological



The component of $\varphi = (x^1, x^2, \dots x^n)$

The map φ is called a (local) coordinate map and the component functions $(x^1, x^2, \dots, x^n)$ of φ defined by

$$\varphi(p) = (x^1(p), \dots x^n(p))$$

are called local coordinates on U.

i, e (U, φ) is a chart containing p.

Lt $(U, (x^1, x^2, \dots x^n))$

or $(U, (x^i(p))$

i,e let M be a topological space and $U \subseteq M$ an open set. let $\hat{U} \subseteq \mathbb{R}^n$ be open.

A homeomorphism of $\phi : U \longrightarrow \hat{U}$ defined by

$$\phi(U) = (x_1(U), x_2(U), \dots, x_n(U))$$

is called a coordinate system on U,

and the functions $x_1, x_2, \dots x_n$ the coordinate functions.

The pair (U, ϕ) is called a chart on M.

Graphs of Continuous functions:

For any open set $U \subset \mathbb{R}^n$ and any continuous map $f : U \longrightarrow \mathbb{R}^n$, the graph of f is the subset in $\mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$

defined by

$$\Gamma(f) = \{(x, y) \mid x \in U, y = f(x)\} \subset \mathbb{R}^{m+n}$$

i.e. $\Gamma(f)$ is the subspace topology of $\mathbb{R}^{m+n}$.

let $\pi_1 : \mathbb{R}^m \times \mathbb{R}^n \longrightarrow \mathbb{R}^m$ denote the projection onto the first coordinates

and let $\varphi : \Gamma(f) \longrightarrow U$ be the restriction of π_1 to $\Gamma(f)$

$$\varphi(x, y) = x, \quad (x, y) \in \Gamma(f)$$

Since π_1 is continuous so φ is continuous

φ is a homeomorphism because φ has continuous inverse given by

$\varphi^{-1}(x) = (x, f(x))$.

Note : $\Gamma(f) = \{(x, y) \in \mathbb{R}^m \times \mathbb{R}^n : x \in U$ and $y = f(x)\}$

$\Gamma(f)$ is Hausdorff and second countable by the theorem product of two second-countable spaces is second-countable.

Thus $\Gamma(f)$ is a topological manifold of dimension n.

In fact, $\Gamma(f)$ is homeomorphic to U itself.

$(\Gamma(f), \varphi)$ is a global coordinate chart.

A global coordinate chart is a coordinate chart whose domain is the whole manifold i.e U can be extended to ~~whole~~ for M where M is manifold.

Also, $\Gamma(f)$ has a single coordinate chart given by $(\Gamma(f), \pi_1)$ where π_1 is projection onto the first coordinates.

A topological space M is a manifold of dimension n if

1. M is Hausdorff,
2. M is second countable and
3. M is locally Euclidean of dimension n.

Example: Unit n-sphere is a topological manifold

$\Rightarrow$ For each $n \geq 0$, the unit n-sphere S^n is Hausdorff and second-countable because it is a topological subspace of $\mathbb{R}^{n+1}$.

Also, unit n-sphere is Hausdorff because

Take northern hemisphere

$$N = \{(x_1, x_2, \dots, x_{n+1}) \in S^n \mid x_{n+1} > 0\}$$

Southern hemisphere

$$S = \{(x_1, x_2, \dots, x_{n+1}) \in S^n \mid x_{n+1} < 0\}$$

Here N and S are open and

$$N \cap S = \phi$$

$\Rightarrow$ S^n is Hausdorff.

Locally Euclidean Space :- let $n \geq 0$ or $n \geq 1$

A topological space X is called locally ~~ett~~ Euclidean space of dimension n if each point of X has a neighborhood homeomorphic either to $\mathbb{R}^n$ or $\mathbb{R}^n_+$.

where $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_1 \geq 0\}$. for $n \geq 1$

Example :-

The unit n-sphere $S^n = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$ is ~~Euclidean~~ locally euclidean

$S^n = U_N \cup U_S$, where $U_N = S^n \setminus \{(0, \dots, 0, 1)\}$ and $U_S = S^n \setminus \{(0, \dots, 0, -1)\}$ are opens for the induced topology of $\mathbb{R}^{n+1}$ on S^n.

~~Now~~ we can also write

$S^n_+ = U_N$ denote ~~ep~~ an open ~~set~~ neighbourhood of the Northpole inside the sphere S^n.

Similarly $S^n_- = U_S$ denote the open neighbourhood of the south pole inside the sphere S^n.

Now using the stereographic projection we have

$$\phi_N : U_N = S^n \setminus \{(0, \ldots, 1)\} \longrightarrow \mathbb{R}^n$$

defined by

$$\phi_N (x_1, x_2, \ldots, x_{n+1}) = \left(\frac{x_1}{1 - x_{n+1}}, \ldots, \frac{x_n}{1 - x_{n+1}} \right)$$

Similarly $\phi_S : U_S = S^n \setminus \{(0, \ldots, -1)\} \longrightarrow \mathbb{R}^n$

$$\phi_S (x_1, \ldots x_{n+1}) = \left(\frac{x_1}{1 + x_{n+1}}, \ldots, \frac{x_n}{1 + x_{n+1}} \right)$$

these show that complement of a point in S^n is homeomorphic to $\mathbb{R}^n$

Note: ① $S^n_+ = \{(x_1, \dots, x_n, x_{n+1}) \in S^n \mid x_{n+1} > 0\}$

$$S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$$

$$\dim(\mathbb{R}^{n+1}) = n+1$$

So $x = (x_1, \dots \dots, x_{n+1})$

② We can define coordinate charts as follow

$$U_k^+ = \{x \in S^n : x_k > 0\}$$

$$\phi_k^+ = (x^1, x^2, \dots, x^{k-1}, \widehat{x^k}, x^{k+1}, \dots, x^{n+1}\}$$

where $\widehat{x}^k$ denote that x^k is not there and x^k is ~~can~~ ommitted

~~to~~ $\Rightarrow \quad \phi_k : U_k \longrightarrow \mathbb{R}^n$

Real Projective Spaces :-

The n- dimensional projective space $\mathbb{RP}^n$ is the set of all lines $l \subseteq \mathbb{R}^{n+1}$.

The quotient space $\mathbb{R}p^n$ is called real projective n-space.

The equivalence class of a point $(x_0, x_1, \ldots, x_n)$ in $\mathbb{R}^{n+1} - \{0\}$ is ~~not~~ denoted by

$$[x_0, x_1, \ldots, x_n]$$

Consider the equivalence relation

$x \sim y \iff$ there exist $\lambda \in \mathbb{R} - \{0\}$

$\Rightarrow x = \lambda y$ on $\mathbb{R}^{n+1} - \{0\}$

i.e for all $x \in \mathbb{R}^{n+1} - \{0\}$

we have $\boxed{x = \lambda y}$

Then $\boxed{\mathbb{R}P^n = (\mathbb{R}^{n+1} \setminus \{0\}) / \sim}$

Also, $\sim$ is the equivalence relation generated by the relation

$$(x_0, x_1, \ldots, x_n) \sim (\lambda x_0, \ldots, \lambda x_n)$$

for any $\lambda \in \mathbb{R} - \{0\}$.

i.e the equivalence class of a point $(x_0, x_1, \ldots, x_n)$ is denoted by

$$[x_0, x_1, \ldots, x_n]$$

To show that $\mathbb{R}P^n$ is homeomorphic to $S^n / \sim$.

A coordinate domain that is homeomorphic ~~to~~ to a ball in $\mathbb{R}^n$ is called a coordinate ball.

or

Consider a topological manifold M

let p be a point of M and U a neighborhood of p homeomorphic to an open ball $B_\varepsilon(x)$ of radius in $\mathbb{R}^n$.

This open ball $B_\varepsilon(x)$ called a coordinate ball.

We know manifold are locally euclidean and euclidean space is locally compact.

$\Rightarrow$ Manifold are locally compact

<u>proof</u>: let $p \in M$. let (U, ϕ) be a chart on topological manifold M and $\phi : U \longrightarrow \hat{U}$ is a homeomorphism from U to an open subset

$\hat{U} = \phi(U) \subseteq \mathbb{R}^n$ for some n.

$\hat{U} = \phi(U)$ is open

Since U is a neighbourhood of point p.

This implies $\phi(p) \subseteq \phi(U)$

So there exist an open ball $B(\phi(p), r)$ with $r > 0$ such that

$$B = B(\phi(p), r) \subset \phi(U)$$

Taking $\bar{B}$ (B closure), then

$$\bar{B} = \overline{B(\phi(p), r)} \subseteq \phi(U)$$

$$\phi(p) \in B$$

$$\phi(p) \in B(\phi(p), r)$$

$$\Rightarrow \phi(p) \in B \Rightarrow p \in \phi^{-1}(B)$$

$$p \in \phi^{-1}(B) \subset \phi^{-1}(\bar{B}) \subseteq U$$

Now by definition of locally ~~compact~~ compact space every point x of X has a compact neighbourhood i.e there exist an open set U and a compact set K, such that

$$x \in U \subseteq K.$$

~~So the~~

So here $p \in \phi^{-1}(B) \subset \overline{\phi^{-1}(B)} \subseteq U$

Since $\overline{\phi^{-1}(B)}$ is compact

Thus M is locally compact.

Refinement:-

let $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ and $V = \{V_\beta\}_{\beta \in J}$

be cover of a space X.

Then V is called a refinement of U if

there exist a map $f : J \longrightarrow I$ such that

$$V_\beta \subset U_{f(\beta)} \text{ for all } \beta \in J$$

Here $f(\beta) \subset I$

We can also say that

$V = \{V_\beta : \beta \in J\}$ is a refinement

of the cover $U = \{U_\alpha : \alpha \in I\}$ if

and only if, for any V_β in V,

there exist some U_α in U such that

$$V_\beta \subseteq U_\alpha.$$

V is called a refinenment of U if for each open covering $\{V_j\} \in V$ of X there exist $\{U_i\} \in U$ with $V_j \subseteq U_i$ i.e if each V_i is contained in some ~~ϵ~~. U_i.

locally finite :- let X be topological space

A collection $\{U_\alpha\}_{\alpha \in I}$ of a space X is called locally finite if each point in X has a neighbourhood V_x such that $V_x \cap U_\alpha \neq \phi$ for only finitely many α.

i.e For any x in X, there exist some neighbourhood $V(x)$ of x such that the set

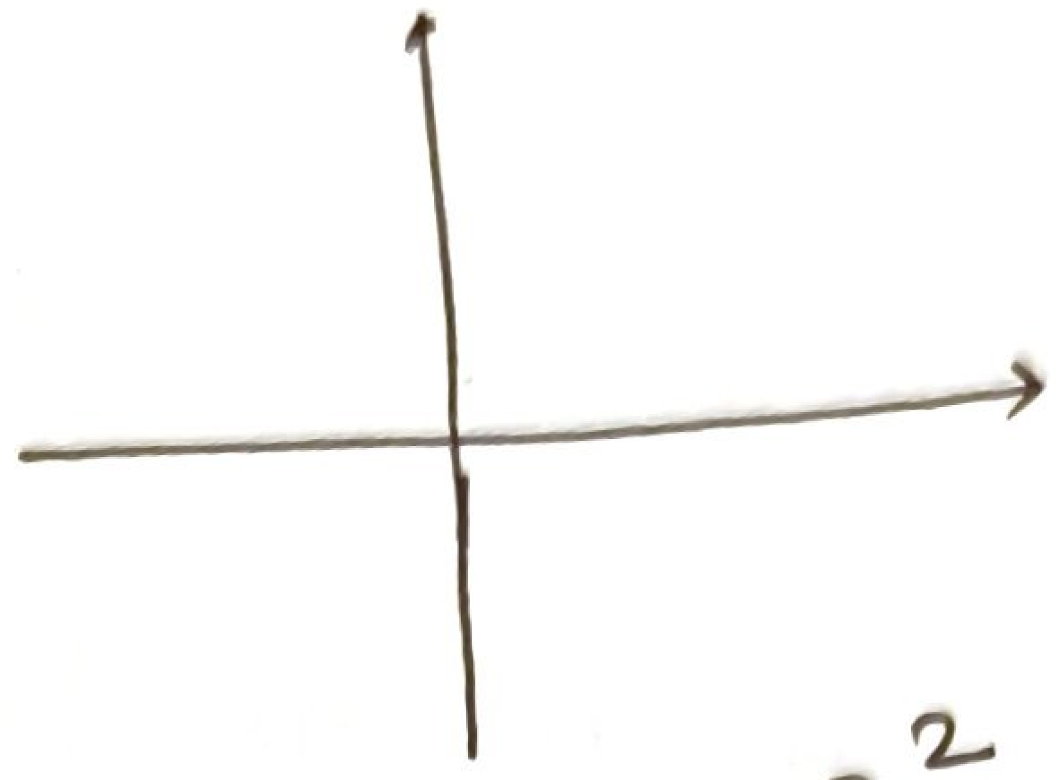
$$\{\alpha \in I : U_\alpha \cap V(x) \neq \phi\}$$ is finite.

$\Rightarrow$ $U_\alpha \cap V_x \neq \phi$ for only a finite number of $i \in I$

Note :- $\{U_\alpha\}_{\alpha \in I}$ also denote open cover.

Paracompact: A topological space X is called paracompact if every open cover of X has an open, locally finite refinement.

Q.



Prove that the cross in $\mathbb{R}^2$ with the subspace topology is not locally Euclidean at p and so can not be a topological manifold.

Sol. let us take M as a cross in $\mathbb{R}^2$ with the subspace topology.

let $B = (0, \varepsilon) \subset \mathbb{R}^n$ be an open ball and p has nbd U. let $p = (0,0)$

then U is any neighbourhood (nbd) of $(0,0)$ in subspace topology.

If M is locally Euclidean, then there is a homomorphism between U to B.

But ~~is~~ $B - (0,0) = B - \{0\}$ is either connected if $n \geq 2$ or has two connected component if $n = 1$.

and $U - \{p\} = U - \{(0,0)\}$

i.e if ~~you~~ we have remove point $(0,0)$

In x, y axis, we have four connected component.

Therefore there can be no homeomorphism from $U - \{p\}$ to $B - \{0\}$.

This contradiction show that cross is not locally Euclidean.

Compatible charts :

Two charts $(U, \phi: U \longrightarrow \mathbb{R}^n)$,

$(V, \psi: V \longrightarrow \mathbb{R}^n)$

or we can write let (U, ϕ) and (V, ψ)

be two charts of the same manifold.

we say that two charts are C^∞- compatible

if the following two map

$$\phi \circ \psi^{-1} : \psi(U \cap V) \longrightarrow \phi(U \cap V)$$

$$\psi \circ \phi^{-1} : \phi(U \cap V) \longrightarrow \psi(U \cap V)$$

are C^∞.

Note: C^∞ and C^K

let K be a non-negative integer. A function

$f: U \longrightarrow \mathbb{R}$ is said to be C^K at p

if its partial derivatives $\dfrac{\partial^j f}{\partial x^{i_1} \cdots \partial x^{i_j}}$ of

all order $j \leq K$ exist and are

continuous at p.

The function $f: U \longrightarrow \mathbb{R}$ is C^∞ at p if it is C^k for all $k \geq 0$.

i.e its partial derivatives of all order

$$\frac{\partial^k f}{\partial x^{i_1} \cdots \partial x^{i_k}}$$ exist and

are continuous at p.

Also, C^∞ means infinitely-differentiable.

Example:- All the ~~polynom~~ polynomial, sine, cosine and exponential function on the real line are all C^∞.

Note :- A C^0 function on U is a continuous function on U.

Let $f: \mathbb{R} \longrightarrow \mathbb{R}$ be $f(x) = x^{1/3}$. Then

$$f'(x) = \begin{cases} \frac{1}{3} x^{-2/3} & \text{for } x \neq 0 \\ \text{underdefined} & \text{for } x = 0 \end{cases}$$

Thus the function f is C^0 but not C^1 at $x = 0$.

$\phi \circ \psi^{-1} : \psi(U \cap V) \longrightarrow \phi(U \cap V)$

$\psi \circ \phi^{-1} : \phi(U \cap V) \longrightarrow \psi(U \cap V)$

These two maps are called the transition functions between the charts.

(*) Transition map / overlap map.

(*) If $U \cap V$ is empty, then the two charts are automatically C^∞-compatible.

$\Rightarrow$ If $U \cap V = \phi$, then $\phi(U \cap V)$ and $\psi(U \cap V)$ are both the empty set in $\mathbb{R}^n$. So they are open in $\mathbb{R}^n$.

i.e ~~$\phi \circ \psi^{-1} : \phi \to \phi$~~

$\phi \circ \psi : \emptyset \to \emptyset$

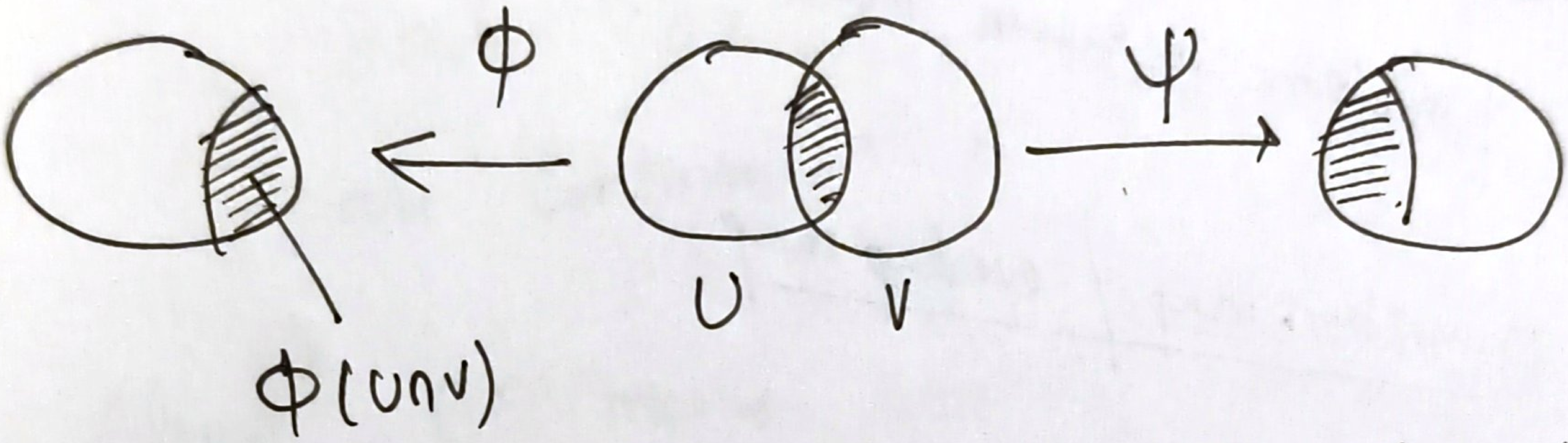
~~$\phi \circ \psi$~~

$\psi \circ \phi^{-1} : \phi \to \phi$

are C^∞ because it is differentiable at every point of ϕ

~~Nota~~ Notation :- $U_{\alpha\beta}$ denote $U_\alpha \cap U_\beta$

$U_{\alpha\beta\gamma}$ denote $U_\alpha \cap U_\beta \cap U_\gamma$



The transition function $\psi \circ \phi^{-1}$ is defined on

$\phi(U\cap V)$

An ~~atlas~~ atlas for a topological space M is an ~~#~~ indexed family ~~$\{(U_\alpha, U_\alpha$~~ $\{(U_\alpha, \phi_\alpha) : \alpha \in I\}$ of charts on M which cover M

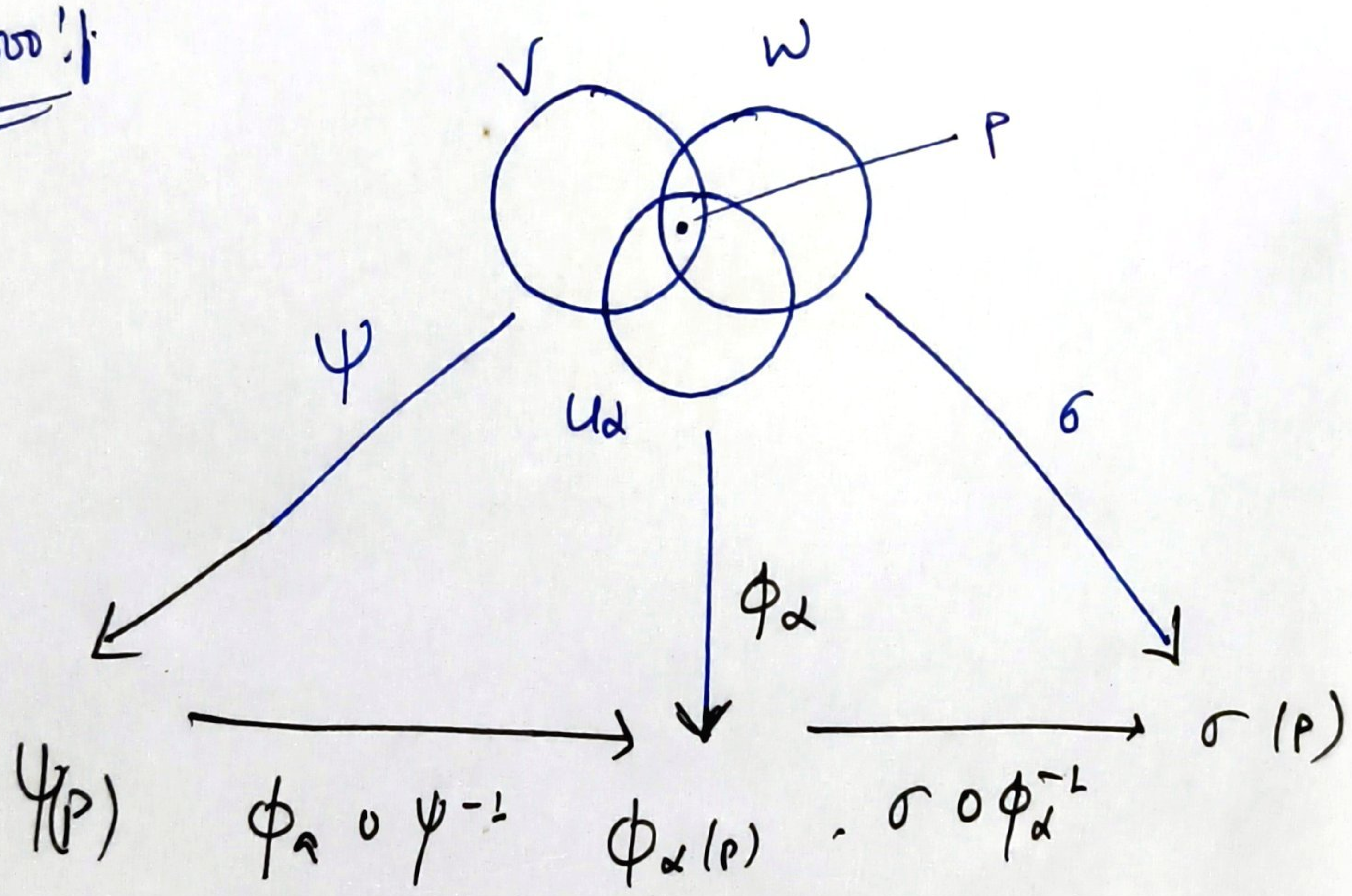
i.e $\bigcup_{\alpha \in I} U_\alpha = M$.

If the co-domain of each chart is the n-dimensional Euclidean space, then M is said to be an n-dimensional manifold.

8

Theorem :- Let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas on locally Euclidean space. If ~~two~~ two charts (V, ψ) and (W, σ) are both compatible with the atlas $\{(U_\alpha, \phi_\alpha)\}$ then they are compatible with each other.

proof :-



Smooth Manifold / C^∞ (infinitely differentiable) manifold :-

let M be a second countable Hausdorff topological space. An n-dimensional smooth atlas on M is a collection of maps

$$A = \{(\varphi_i, U_i) \mid i \in I\}, ~~\varphi_i : U_i \to \varphi_i(U_i) \subset \mathbb{R}^n~~$$

where $\varphi_i : U_i \longrightarrow \varphi_i(U_i) \subset \mathbb{R}^n$ such that

all $U_i \subset M$ are open, all φ_i are homeomorphisms and

$\{U_i ; i \in I\}$ is an open covering of M

$\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \longrightarrow \varphi_i(U_i \cap U_j)$

are smooth for all $i, j \in I$

Note :- I is an index set

→ A smooth atlas on X is called maximal if it is not contained in any other atlas.

i.e A is a maximal C^k - atlas of dimension d if and only if A is not strictly contained in another C^k - atlas.

(*) Two atlases $A = \{ \varphi_\alpha : V_\alpha \longrightarrow U_\alpha \}_{\alpha \in I}$

$B = \{ \varphi_\beta ; V_\beta \longrightarrow U_\beta' \}_{\beta \in I'}$

where I and I' are index set.

Now atlas A and B on X are compatible if their union is itself an ~~atlas~~ atlas.

Smooth structure on a topological manifold is given by a smooth atlas of coordinate ~~char~~ charts, i.e the transition functions between the coordinates charts are C^∞ smooth.

⇒ A smooth structure is used to define differentiability of real value function on a manifold.

Note:- A smooth structure is a maximal C^∞ atlas.

+ C^k structure (differentiable) ~~struct~~ structure also called smooth structure

we can also says that C^k structure on a manifold M is exactly a maximal atlas A with all transition function of class C^k.

Smooth structure on a topological manifold is given by a smooth atlas of coordinate ~~char~~ charts, i.e the transition functions between the coordinates charts are C^∞ smooth.

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Let $M = \mathbb{R}$ and let A_1, A_2 be two atlases on M each consisting of a single map $A_1 = \{\varphi_1\}$, $A_2 = \{\varphi_2\}$

where $\varphi_i : \mathbb{R} \longrightarrow \mathbb{R}$ $(i = 1, 2)$ are given by $\varphi_1(x) = x$

$\varphi_2(x) = x^3$

These atlases are not compatible because $\varphi_2^{-1} \circ \varphi_1(x) = \sqrt[3]{x}$ is not smooth at 0.

~~ie~~ since $\left(\varphi_2(x)\right)^{1/3} = \varphi_2(x^{1/3}) = x$

$\varphi_2^{-1} \circ \varphi_1(x)$

$\varphi_2^{-1} \circ x$ ~~and~~ $\Rightarrow$ $\varphi_2^{-1}(x)$

We know that $\varphi_2(x) = x^3$

$x^{1/3} = \varphi_2^{-1}(x)$

N- Dimensional smooth atlas on M

let M be a second countable Hausdorff topological space.

~~Let~~ An n-dimensional smooth atlas on M ~~is~~ ~~the~~ a collection of maps

$\mathcal{A} = \{(\varphi_i, U_i) \mid i \in A\}$ where

$\varphi_i : U_i \longrightarrow \varphi_i(U_i) \subset \mathbb{R}^n$

Such that ~~f~~ all $U_i \subset M$ are open,

all φ_i are homeomorphisms, and

$\{U_i, i \in A\}$ is an open covering of M

$\varphi \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \longrightarrow \varphi_i(U_i \cap U_j)$

are smooth for all $i, j \in A$.

$\Rightarrow (\varphi_i, U_i), i \in A$ are called charts on M

$\varphi_i \circ \varphi_j^{-1}$, whenever defined are called transition function.

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$\Rightarrow (\varphi_i, U_i), i \in A$ are called charts on M

$\varphi_i \circ \varphi_j^{-1}$, whenever defined are called transition function.

Theorem :- Let M be a topological manifold. Every smooth atlas for M is contained in a unique maximal smooth atlas.

proof: Let $\mathcal{A}$ be a smooth atlas for M and let $\bar{\mathcal{A}}$ denote the set of all charts that smoothly compatible with every chart in $\mathcal{A}$.

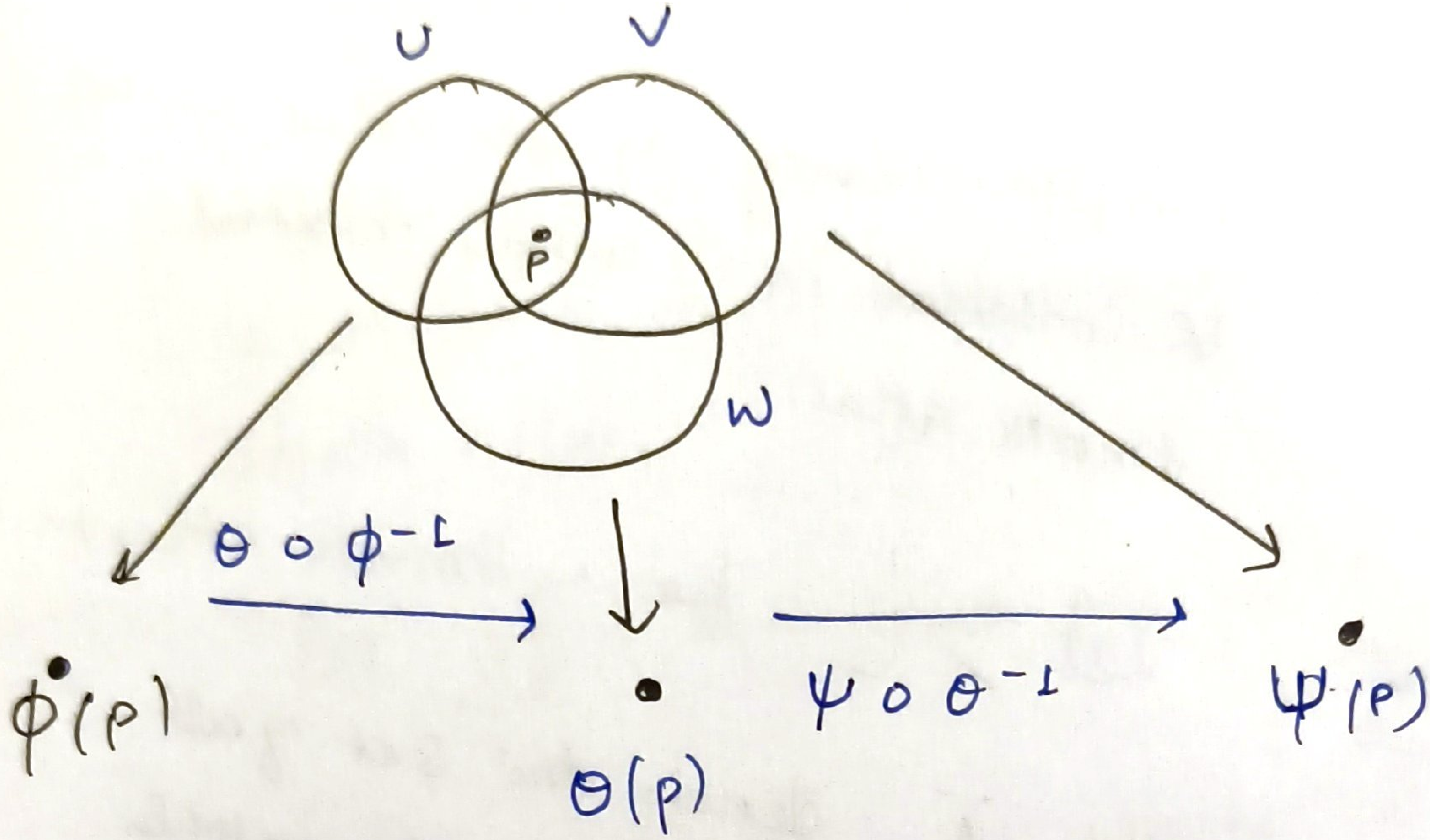
To show :- $\bar{\mathcal{A}}$ is a smooth atlas.

proof :- Let $(U, \phi), (V, \psi)$ ~~are~~ be two chart in $\bar{\mathcal{A}}$.

Let ~~$p \in V \cap W$~~ $p \in U \cap V$.

Since $\mathcal{A}$ is a~~tlas~~ smooth ~~a~~ atlas, so there is some $(W, \theta) \in \mathcal{A}$ ~~and $p \in W$~~ such that $p \in W$.

Then $p \in U \cap V \cap W$.



By definition of $\bar{A}$, both $\theta \circ \phi^{-1}$ and $\psi \circ \theta^{-1}$ are smooth, because every transition map is smooth.

Since $p \in U \cap V \cap W$, then

$$\psi \circ \phi^{-1} = (\psi \circ \theta^{-1}) \circ (\theta \circ \phi^{-1})$$

is smooth on a nbd of $\phi(p)$

we

we can also say that

$$\psi \circ \phi^{-1} = (\psi \circ \theta^{-1}) \circ (\theta \circ \phi^{-1})$$

is C^∞ on $\phi(V \cap W \cap U)$

Hence $\psi \circ \phi^{-1}$ is smooth on a nbd of each point in $\phi(U \cap V)$.

Therefore $\bar{A}$ is a smooth atlas.

To show that $\bar{A}$ is maximal. and unique.

proof:- Suppose $\bar{A}$ is contained in another atlas, A'. Then clearly A is contained in A', and so all charts in A' are compatible with all chart in A.

Since any chart that is smoothly compatible with every chart in $\bar{A}$ is automatically smoothly compatible with every chart in A.

Thus, A' is contained in $\bar{A}$.

It follows that $\bar{A} = A'$

Now suppose that there is another maximal atlas A'' containing A.

Since every chart in A'' is compatible with every chart in A, we must have that $A'' \subseteq \bar{A}$

Since A'' is maximal, it follows that

$$\boxed{A'' = \bar{A}}$$

$GL_n(\mathbb{R})$ is a manifold

proof:- Let $M_n(\mathbb{R})$ be the set of all $n \times n$ matrices.

Define a map

$$f : M_n(\mathbb{R}) \longrightarrow \mathbb{R} \text{ by}$$

$$f(A) = \det A \quad \text{where } A \in M_n(\mathbb{R})$$

We also know that $\det A$ is continuous function

because $\det A$ is a polynomial in the coordinates

$$\det(a_{ij}) = \sum_{\sigma \in S_n} sg(\sigma) \prod_{i=1}^{n} a_{i\sigma i}$$

i.e. $\det(A)$ is a ~~polynomials~~ polynomial in the variable $a_{i,\sigma i}$

Take $n=3$

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix}_{n \times n}$$

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix}$$

$$\det(A) = \sum_{\sigma \in S_n} sgn(\sigma) \prod_{i=1}^{n} a_{i,\sigma_i}$$

$$= Sgn([1 2 3]) \prod_{i=1}^{n} a_{1} [1,2,3]_i + \cancel{sgn([1,3,2]} \prod$$

$$Sgn[(1,3,2]) \prod_{i=1}^{n} a_i [1,3,2]_i$$

$$+ Sgn([2,1,3]) \prod_{i=1}^{n} a_i, [2,1,3]_i$$

$$+ sgn([2,3,1]) \prod_{i=1}^{n} a_i [2,3,1]_i \quad \cancel{+ sgn([3,1,2]]}$$

$$+ sgn[3,1,2] \prod_{i=1}^{n} a_i [3,1,2]_i$$

$$+ \ \mathrm{Sgn}([3,2,1]) \prod_{i=1}^{n} a_{i,[3,2,1]_i}$$

$$= \prod_{i=1}^{n} a_{i,[1,2,3]} - \prod_{i=1}^{n} a_{i\,[1,3,2]_i}$$

$$- \prod_{i=1}^{n} a_{i,[2,1,3]}$$

$$+ \prod_{i=1}^{n} a_{i,[2,3,1]_i} + \prod_{i=1}^{n} a_{i,[3,1,2]_i}$$

$$- \prod_{i=1}^{n} a_{i,[3,2,1]_i}$$

$$= a_{1,1} a_{2,2} . a_{3,3} - a_{1,1} a_{2,3} . a_{3,2}$$

$$- a_{1,2} a_{2,1} a_{3,3} + a_{1,2} a_{2,3} a_{3,1}.$$

$$+ a_{1,3} a_{2,1} a_{3,2} - a_{1,3} a_{2,2} a_{3,1}$$

Now if $A \in GL_n(\mathbb{R}) \subseteq M_n(\mathbb{R})$

then $f(GL_n(\mathbb{R})) = \mathbb{R} - \{0\}$.

Since $GL_n(\mathbb{R}) = \{A \in \mathbb{R}^{n \times n} \mid \det A \neq 0\}$

$$f^{-1}(\mathbb{R} - \{0\}) = GL_n(\mathbb{R})$$

Also, $\mathbb{R} - \{0\}$ is an open subset of $\mathbb{R}$.

$\Rightarrow$ $GL_n(\mathbb{R}) = \det^{-1}(\mathbb{R} - \{0\})$ is an open subset of $\mathbb{R}^{n^2} \cong \mathbb{R}^{n \times n}$

An open subset of a smooth manifold is itself a smooth manifold.

Therefore $GL_n(\mathbb{R})$ is a manifold.

~~M-dim~~ m - dimensional manifold :-

An m- dimensional manifold is a set M, together with a countable collection of subset $U_\alpha \subset M$, called coordinates charts.

And injective functions $X_\alpha : U_\alpha \longrightarrow V_\alpha$ onto connected open subsets V_α of $\mathbb{R}^m$ called local coordinates, which satisfy the following properties

(1) The coordinate chart cover M.

$$\bigcup_\alpha U_\alpha = M$$

(2) On the overlap of any pair of coordinate charts $U_\alpha \cap U_\beta$ the composite map

$$X_\beta \circ X_\alpha^{-1} : X_\alpha (U_\alpha \cap U_\beta) \to X_\beta (U_\alpha \cap U_\beta)$$

is a smooth function.

(4) ~~If $x \in U_\alpha$, $y \in U_\beta$ are disjoint~~

(3) if $x \in U_\alpha$, $y \in U_\beta$ are distinct points of M, then there exist open subsets W of $X_\alpha(x)$ in V_α and Z of $X_\beta(y)$ in V_β such that

$$X_\alpha^{-1}(W) \cap X_\beta^{-1}(Z) = \phi$$

Unit circle is a one-dimensional Manifold :-

$$S^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}.$$

Chart (U_1, X_1) and (U_2, X_2) for S^1 may be define as follow :-

Here $U_1 = S^1 \setminus \{(0,1)\}$

$U_2 = S^1 \setminus \{(0,-1)\}$

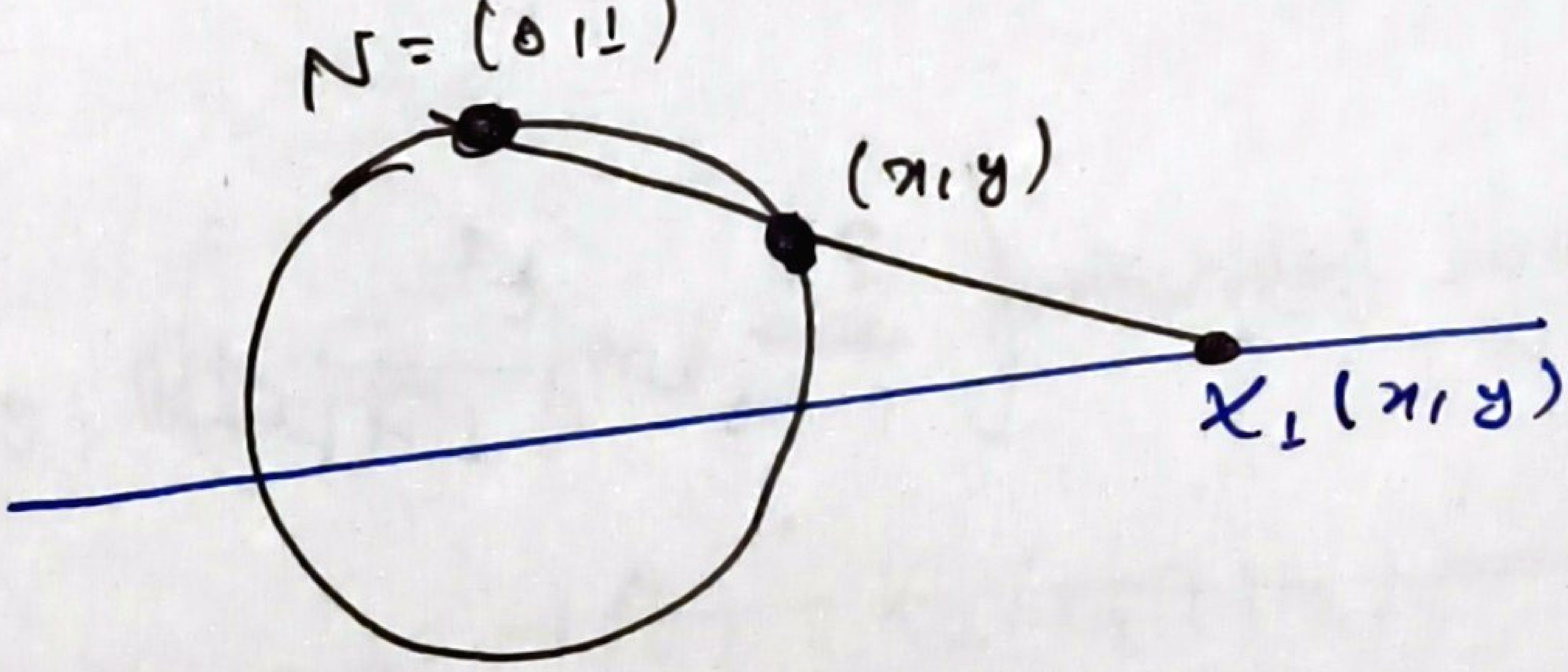
i.e the unit circle with deleted north and south pole respectively.

We then let the coordinate map

$$X_1 : U_1 \longrightarrow \mathbb{R} \ , \quad X_2 : U_2 \longrightarrow \mathbb{R}$$

where X_1 be the stereographic projection from north pole

X_2 be the stereographic projection from south pole.



$$X_1(x,y) = \left(\frac{x}{1-y}\right)$$

$$X_2(x,y) = \left(\frac{x}{1+y}\right).$$

$$X_1^{-1} : \mathbb{R} \longrightarrow U_1$$

$X_1^{-1}(t) = \cancel{t}$

Solving for t in $X_2(x,y) = t$

put $\frac{x}{1+y} = t$

find the value x using $x^2+y^2=1$ equation.

finally we have,

$$X_2^{-1}(t) = \left(\frac{2t}{t^2+1}, \frac{1-t^2}{1+t^2} \right)$$

similarly $X_1^{-1}(t) = \left(\frac{2t}{1+t^2}, \frac{t^2-1}{1+t^2} \right)$

let $(x,y) \in U_1 \cap U_2$

then $(0,-1) \neq (x,y)$

$(0,1) \neq (x,y)$

since $U_1 = S^1 - \{(0,+1)\}$

$U_2 = S^1 - \{(0,-1)\}$

$$X_1(x,y) = \frac{x}{1-y} \neq 0 \quad \text{because } x \neq 0$$

$$\Rightarrow X_1 : U_1 \cap U_2 \longrightarrow \mathbb{R} - \{0\}$$

$$X_2(x,y) = \frac{x}{1+y} \neq 0$$

$$\Rightarrow X_2 : U_1 \cap U_2 \longrightarrow \mathbb{R} - \{0\}$$

finally, we have $X_1(U_1 \cap U_2) = \mathbb{R} - \{0\}$

$X_2(U_1 \cap U_2) = \mathbb{R} - \{0\}$

Now, let us find the transition function

$X_1 \circ \cancel{X_2^{-1}(t) : X_2(U_1 \cap U_2) \longrightarrow X_1(U_1 \cap U_2)}$

$$X_1 \circ X_2^{-1} : X_2(U_1 \cap U_2) \longrightarrow X_1(U_1 \cap U_2)$$

$$X_1 \circ X_2^{-1} : \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R} \setminus \{0\} \text{ is}$$

a smooth map.

Now, let us find the transition function

let $t \in \mathbb{R} - \{0\}$

$$X_1 \circ X_2^{-1}(t) = X_1\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)$$

$$= \frac{\frac{2t}{1+t^2}}{1 - \frac{(1-t^2)}{(1+t^2)}}$$

$$= \frac{\frac{2t}{1+t^2}}{\frac{2t^2}{1+t^2}}$$

$$= \frac{1}{t}$$

~~$X_2 \circ X_1$~~ $X_1 \circ X_2^{-1}(t) = 1/t$

which is clearly a smooth map away from the origin.

Since $\mathbb{R}^2$ is Hausdorff.

$\Rightarrow$ ~~X~~ $X_1^{-1} \cap X_2^{-1} = \phi$

Therefore unit circle is a one-~~dimen~~ dimensional Manifold.