Norm spaces

(1)

let $(X, +, \cdot)$ be a vector space over $\mathbb{F}$.

A norm on X is a function form X to $[0, \infty)$

satisfying the following 3 properties

$\forall\ x, y \in X$ and $\forall$ ~~$\lambda \in \mathbb{F}$~~ $\lambda \in \mathbb{F}$

(1) $\|x\| \geq 0$, and if $x = 0$ then $\|x\| = 0$

(2) $\|x + y\| \leq \|x\| + \|y\|$

(3) $\|\lambda x\| = |\lambda| \|x\|$

Example : (1) $(\mathbb{C}^n, \|\cdot\|_p)$, $1 \leq p < \infty$

where $\|x\|_p = \left(\sum_{k=1}^{n} |x_k|^p \right)^{1/p}$

(2) $(\mathbb{F}^n, \|\cdot\|_\infty)$ where

$$\|x\|_\infty = \max_{k=1,\dots,n} |x_k|$$

$\|x\|_\infty$ denotes supnorm.

(3) let l^p, $1 \le p < \infty$, be the collection

of all F-valued sequences

$x = (x_1, x_2 \ldots)$ satisfying (2)

$$\sum_{k=1}^{\infty} |x_k|^p < \infty$$

where

$$\|x\|_p = \left(\sum_{k=1}^{\infty} |x_k|^p \right)^{1/p}$$

(4) $(C[a,b], \|\cdot\|_\infty)$

where $\|f\|_\infty = \sup_{x \in [a,b]} |f(x)|$

(5) $(C[a,b], \|\cdot\|_p)$

where $\|f\|_p = \left(\int_a^b |f(x)|^p dx \right)^{1/p}$

Cauchy Sequence in Normed space.

$\Rightarrow$ let X be a normed space and let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence elements of X.

① $\{f_n\}_{n \in \mathbb{N}}$ converges to $f \in X$

if $\lim\limits_{n \to \infty} \|f - f_n\| = 0$ i.e

if $\forall\ \varepsilon > 0$, there exist $(\exists)$ $N > 0$, for all $(\forall)$ $n \geq N$, $\|f - f_n\| < \varepsilon$.

i.e $\lim\limits_{n \to \infty} f_n = f$ $\Rightarrow \|f_n - f\| < \varepsilon$ or $\|f - f_n\| < \varepsilon$.

② $\{f_n\}_{n \in \mathbb{N}}$ is ~~Cau~~ Cauchy if

$\forall\ \varepsilon > 0$, there exist $N > 0$, for all $m, n \geq N$, $\|f_m - f_n\| < \varepsilon$.

If $(X, \|\cdot\|)$ is a normed vector space,

then function $d : X \times X \longrightarrow \mathbb{R}$ (4)

defined by ~~d(x,y) = \|x-y~~

$d(x,y) = \|x - y\|$ is a metric

on X

Some special properties

(1) Homogeneity mean $d(\lambda x, \lambda y) = |\lambda| d(x,y)$

~~(2) Translation invariance~~

(2) Translation invariance : $d(x+z, y+z) = d(x,y)$

Notes :- $d(x,0) = \|x - 0\| = \|x\|$

Absolute convergent of Normed space

(#) let $(X, \|\cdot\|)$ be a normed space.

$\sum_{n=1}^{\infty} x_n$ is absolutely convergent if

$$\sum_{n=1}^{\infty} \|x_n\| < \infty$$

(*) properties of Cauchy sequence in Normed space

(1) $(\mathbb{F}^n, \|\cdot\|_\infty)$ where $\|x\|_\infty = \max_{k=1,2\dots n} |x_k|$

Here let $\mathbb{F}^n$ denote the vector space of sequence of real numbers that become 0 (zero) after nth terms,

$$X_n = (\underbrace{1, \tfrac{1}{2}, \tfrac{1}{3} \dots \dots \tfrac{1}{n}}_{n\text{-times}}, 0, 0, \dots \dots)$$

Here m entry of X_n is $\frac{1}{m}$ when $m \le n$

m entry of X_n is 0 when $m > n$.

Now ~~to show~~ show that $\{X_n\}_{n=1}^{\infty}$ is a cauchy sequence

(6)

proof :- Step 1: Take any $\varepsilon > 0$ and then choose some $N > 1/\varepsilon$.

Step 2. If $n, m > N$, then we assume $m > n$, Now we have.

$$\|x_n - x_m\|_\infty = \sup \| (1, \tfrac{1}{2}, \cdots \tfrac{1}{n+1}, \tfrac{1}{n}, 0 \cdots 0)$$

$$- (1, \tfrac{1}{2}, \tfrac{1}{3}, \cdots, \tfrac{1}{m+1}, \tfrac{1}{m}, 0 \cdots) \|$$

$$= \sup \| (0, 0, \cdots \cdots 0, \cdots, -\tfrac{1}{m+1}, \tfrac{-1}{n}, \cdots - \tfrac{1}{m}, 0 \cdots 0) \|$$

$$= \sup \| (0, 0, \cdots, 0, -\tfrac{1}{m+1}, \tfrac{-1}{n}, \cdots, -\tfrac{1}{m}, 0 \cdots 0 \|$$

$$= \frac{1}{n+1} < \frac{1}{N} < \varepsilon.$$

$$\Rightarrow \boxed{\|x_n - x_m\| < \varepsilon}$$

So, $\{x_n\}_{n=1}^{\infty}$ is a cauchy sequence.

$\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence

but $\{x_n\}_{n=1}^{\infty}$ is not convergent

(7)

in $(\mathbb{F}^n, \|\cdot\|_\infty)$

because

$$x_m^n = \begin{cases} \frac{1}{m} & \text{for } m \le n \\ 0 & \text{if } m > n. \end{cases}$$

i.e $\lim_{n\to\infty} x_n \neq x$

$$\|x_n - x\| = \left\|\frac{1}{m} - 0\right\|$$
$$= \frac{1}{m} \neq 0$$

(2) $(\mathbb{F}^n, \|\cdot\|_2)$

where $\|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}$

Take $x_n = \left(1, \frac{1}{2}, \frac{1}{3}, \dots \frac{1}{n}, 0 \dots 0\right)$

Here ~~the~~ x_n is Cauchy sequence in $(\mathbb{F}^n, \|\cdot\|_2)$

$\| x_n - x_m \| = \| (0, \ldots 0, -\frac{1}{n+1}, \ldots -\frac{1}{m}, 0, \ldots) \|_2$

$$= \left(\sum_{k=n+1}^{m} \frac{1}{k^2} \right)^{1/2} < \left(\sum_{k=n+1}^{\infty} \frac{1}{k^2} \right)^{1/2}.$$

(8)

we know $\sum \frac{1}{k^2}$ is ~~a~~ convergent.

so $\left(\sum_{k=n+1}^{\infty} \frac{1}{k^2} \right)^{1/2} < \varepsilon$.

So X_n is a cauchy sequence, but X_n is not convergent in $(\mathbb{F}^n, \|\cdot\|_2)$

$\| X_n - X \|_2 = \| 0, 0, 0 \ldots, \frac{1}{m}, \ldots 0 \|_2$

$$= \left(0 + \ldots + \frac{1}{m^2} + \ldots \right)^2$$

$\neq 0$

$$\boxed{\lim_{n \to \infty} X_n \neq X}$$

Banach spaces :-

A normed space X is called a Banach space if every Cauchy sequence is convergent in X. 9

i.e $\{f_n\}_{n \in \mathbb{N}}$ is Cauchy in X.

and $\{f_n\}$ must be convergent i.e

$f_n \longrightarrow f$.

$$\boxed{\text{Cauchy sequence} + \underset{\text{uniformly}}{\text{Convergent}} = \text{Banach space}}$$

or.

A normed space $(\mathbb{F}, \|\cdot\|)$ is called a Banach space if $\mathbb{F}$ is complete metric space by using given norm $\|\cdot\|$.

Example :- $\left(C[a,b], \| \cdot \|_\infty \right)$ is a banach space

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proof:- $C[a,b]$ meaning $f : [a,b] \to \mathbb{R}$

$C[a,b]$ is a complete metric because

$\{f_n(x)\}$ is cauchy in $\mathbb{R}$ for each $x \in [0,1]$

since Both $\mathbb{R}$ and $[0,1]$ are complete.

proof:- $\|f\|_\infty = \sup_{x \in [a,b]} |f(x)|$

if $\langle f_n \rangle$ is cauchy, then for any $\varepsilon > 0$,

there exist N such that $n, m \geq N$

$$\sup_{x \in [a,b]} |f_n(x) - f_m(x)| < \varepsilon.$$

i,e $\langle f_n \rangle$ is cauchy for each $x \in [a,b]$.

$$f_n : [a,b] \longrightarrow \mathbb{R}$$

$$\lim_{n \to \infty} f_n(x) = f(x)$$

$$\lim_{n \to \infty} f_n(x) = f(x)$$

It satisfy the property.

$$\boxed{\text{Cauchy sequence} + \text{Convergent uniformly} = \text{Banach space}}$$

But $(C[a,b], \|\cdot\|_1)$ and $(C[a,b], \|\cdot\|_2)$ are not Banach space.

$(C[a,b], \|\cdot\|_1)$ is not Banach space

proof:- Because $C[a,b]$ with $\|\cdot\|_1$ is not complete metric space.

Take $d_1(f,g) = \int_0^1 |f(x) - g(x)|\, dx$

or $\|\cdot\|_1 = \int_0^1 |f(x)|\, dx$.

$f_n : [0,1] \longrightarrow \mathbb{R}$.

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$$f_n(x) = \begin{cases} 0, & \text{if } 0 \le x \le \frac{1}{2} - \frac{1}{n} \\ nx - \frac{n}{2} + 1 & \text{if } \frac{1}{2} - \frac{1}{n} \le x \le \frac{1}{2} \\ 1 & \text{if } x \in [\frac{1}{2}, 1] \end{cases}$$

i.e $\frac{1}{2} \le x \le 1$.

$\langle f_n(x) \rangle$ is Cauchy sequence because.

$$d_1(f_m, f_n) = \| f_m - f_n \|$$

$$= \int_0^1 f_m \, dx - \int_0^1 f_n(x) \, dx$$

$$= \int_0^{\frac{1}{2} - \frac{1}{m}} f_m \, dx + \int_{\frac{1}{2} - \frac{1}{m}}^1 f_m \, dx - \left[\int_0^{\frac{1}{2} - \frac{1}{n}} f_n \, dx + \int_{\frac{1}{2} - \frac{1}{n}}^1 f_n \, dx \right]$$

$$= 0 + \int_{\frac{1}{2} - \frac{1}{m}}^1 f_m \, dx - \left[0 + \int_{\frac{1}{2} - \frac{1}{n}}^1 f_n \, dx \right]$$

$$= \int_{\frac{1}{2}-\frac{1}{m}}^{1} f_m(x)\,dx - \int_{\frac{1}{2}-\frac{1}{n}}^{1} f_n(x)\,dx.$$

(13)

$$= \frac{1}{2m} - \frac{1}{2n}$$

$$= \frac{1}{2}\left(\frac{1}{m} - \frac{1}{n}\right)$$

Now if $m \to \infty,\ n \to \infty$

$d_1(f_m, f_n) = 0 < \varepsilon.$

$d_1(f_m, f_n) = 0$

so $\langle f_n(x) \rangle$ is a Cauchy sequence in $(C[a,b], \|\cdot\|_1)$

But $\langle f_n(x) \rangle$ is not uniformly convergent

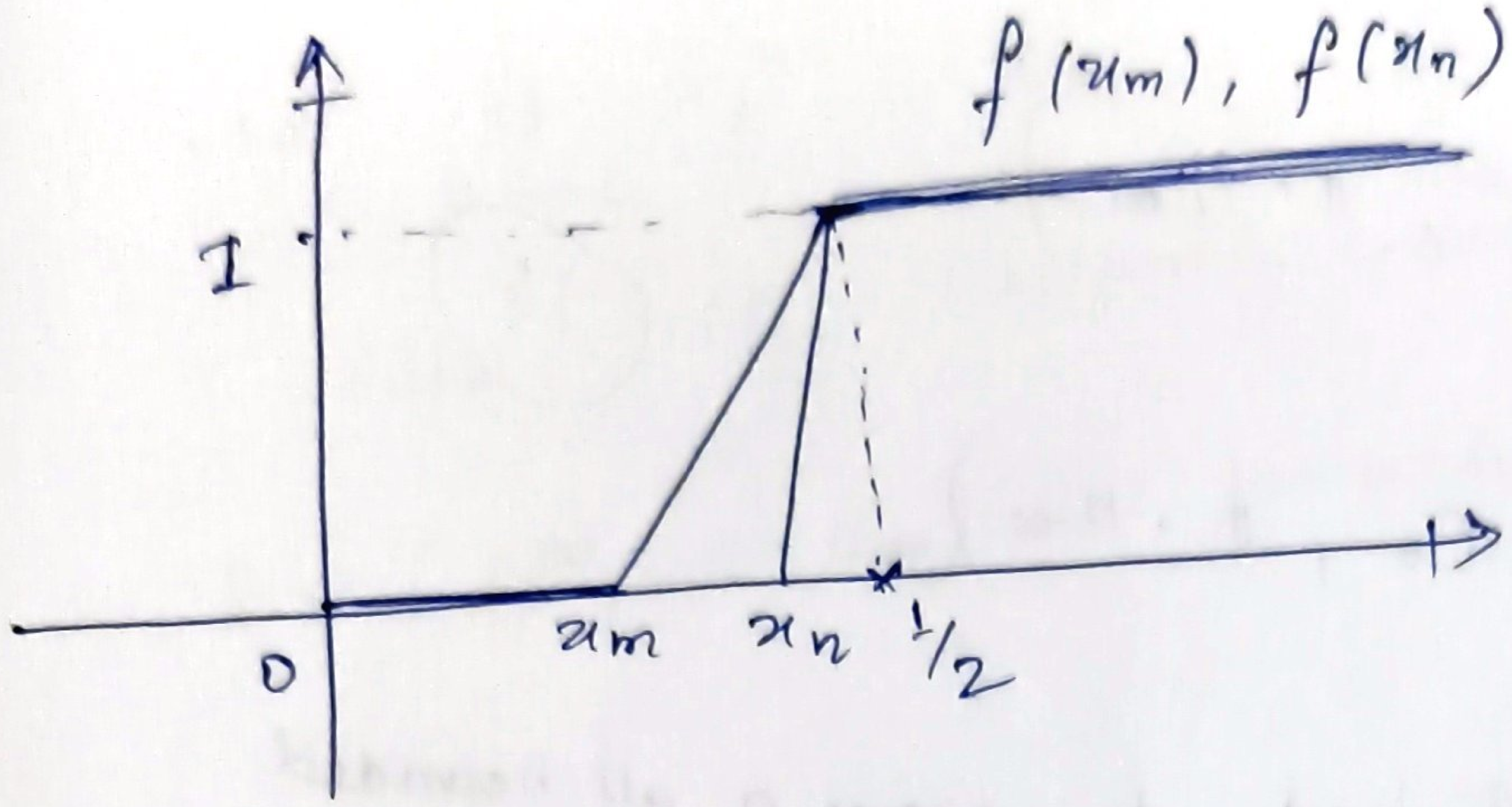
because. $f_n(x)$ is converging to discontinuous function

$$\lim_{n\to\infty} f_n(x) = f(x) = \begin{cases} 0 & \text{if } x \in [0, \tfrac{1}{2}]. \\ 1 & \text{if } x \in [\tfrac{1}{2}, 1] \end{cases}$$

$f(x)$ is not continuous, so not convergent

i.e $(C[a,b], \|\cdot\|_1)$ is not a Banach space.

graph of $f_n(x)$



$n > m$.

$$x_n = \frac{1}{2} - \frac{1}{n}$$

$$x_m = \frac{1}{2} - \frac{1}{m}.$$

From graph
Here, $f(x_m)$, and $f(x_n)$ are not uniformly convergents

so $(C[a,b], \|\cdot\|_1)$ is not a banach space

similarly $(C[a,b], \|\cdot\|_2)$ is also not banach space (Same logic in $\|\cdot\|_1$)

Some other Example of Banach space

are $(\ell^{\infty}, \|\cdot\|_{\infty})$

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and $[c_0, \|\cdot\|_{\infty})$

where ℓ^{∞} denote the space of all bounded sequence of real number $(x_n)_{n=1}^{\infty}$

$$c_0 = \{(x_n)_{n=1}^{\infty} \in \ell^{\infty} : \lim_{n\to\infty} x_n = 0\}$$

<u>proof:</u> $x^n = (x_1^n, x_2^n, \dots)$

$\{x^n\}_{n\in n}$ is a cauchy sequence in ℓ^{∞}.

$$\|x^n - x^m\|_{\infty} < \varepsilon.$$

i/e $\sup_{k=1}^{\infty} |x_k^n - x_k^m| < \varepsilon.$

real number ($\mathbb{R}$) is complete,

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so $\lim\limits_{n \to \infty} x_k^n = x_k$

i.e it uniformly convergent in $\mathbb{R}$.

So $(\ell^\infty, \|\cdot\|_\infty)$ is banach space.

$C_0 = \{(x_n)_{n=1}^{\infty} \in \ell^\infty : \lim\limits_{n \to \infty} x_n = 0\}$ is

closed in ℓ^∞

so $(C_0, \|\cdot\|_\infty)$ is also Banach space.

i.e $\|x^n - x^m\|_\infty < \varepsilon$

$\lim\limits_{n \to \infty} \sup\limits_{k=1}^{\infty} |x_k^n - x_k^m| = 0$

Inner product space / Pre-hilbert space.

Let V be vector space over a field $\mathbb{F}$. (17)

Now the complex valued function

$f = \langle \cdot, \cdot \rangle : V \times V \longrightarrow \mathbb{F}$ is given

by $f(u, v) = \langle u, v \rangle$

that arrange for each $(u, v) \in V \times V$

a value in $\mathbb{F}$.

Now f is an innerproduct if satisfies the given following properties

① $f(u, v) = \langle u, v \rangle = \overline{\langle v, u \rangle}$

$\Rightarrow$ Hermitian - symmetric property.

② <u>Additivity in First argument</u>

$f(u + u', v) = \langle u, v \rangle + \langle u', v \rangle$

$= \langle u + u', v \rangle$

(3) additivity in 2nd argument.

$$f(u, v+v') = \langle u, v+v' \rangle$$
$$= \langle u, v \rangle + \langle u, v' \rangle.$$

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(4) positivity : $f(u,u) = \langle u, u \rangle$

if $\langle u,u \rangle \geq 0$ and $\langle u,u \rangle = 0 \Rightarrow u = 0$.

(5) Homogeneous :-

$$f(\lambda u, v) = \langle \lambda u, v \rangle = \lambda \langle u, v \rangle.$$

if $\mathbb{F} = \mathbb{C}$

$$f(u, \lambda v) = \bar{\lambda} \langle u, v \rangle$$

(*) Note :- $f(u,v)$ (not $f\langle u,v \rangle$)

$f(u,v)$ ✓ (right), $f\langle u,v \rangle$ ✗ (wrong)

$f(u,v)$ is continuous function with respect to

$$\langle \cdot, \cdot \rangle$$

Euclidean Inner product space / unitary space

$\left(\mathbb{F}^n, \langle \cdot, \cdot \rangle\right)$ with

$$f(x,y) = \langle x, y \rangle = \sum_{k=1}^{n} x_k \overline{y_k}$$

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if $\mathbb{F}^n = \mathbb{R}^n$ (~~Sub~~ Euclidean Space)

$\mathbb{F}^n = \mathbb{C}^n$ (unitary space.

Cauchy - Schwarz inequality :

For any x and y in an inner product space $(X, \langle \cdot, \cdot \rangle)$

$$\boxed{|\langle x, y \rangle| \leq \sqrt{\langle x, x \rangle} \sqrt{\langle y, y \rangle}}$$

(Cauchy - Buniakowski - Schwarz theorem) :-

If $u, v \in \mathbb{R}^n$, then $\boxed{|u \cdot v| \leq \|u\| \|v\|}$ — (1)

the given inequality hold exactly when one vectors is a scalar multiple of the other.

(20)

proof:- Take $f(t) = \|tu + v\|^2$

we know that $f(t) \geq 0$ for all $t \in \mathbb{R}$

Because f is defined / continuous on $[0, \infty)$

Suppose $f(t) = 0$ for some $t \in \mathbb{R}$.

i.e $\|tu + v\|^2 = 0 \Rightarrow v = -tu$

i.e inequality hold exactly when one vector is a scalar multiple of the other.

$$\|tu + v\|^2 = (tu + v) \cdot (tu + v)$$

$$= t^2 u \cdot u + 2tu \cdot v + v \cdot v$$

$$= \|u\|^2 \cdot t^2 + (2u \cdot v)t + \|v\|^2$$

Take discriminant $D = \sqrt{b^2 - 4ac}$

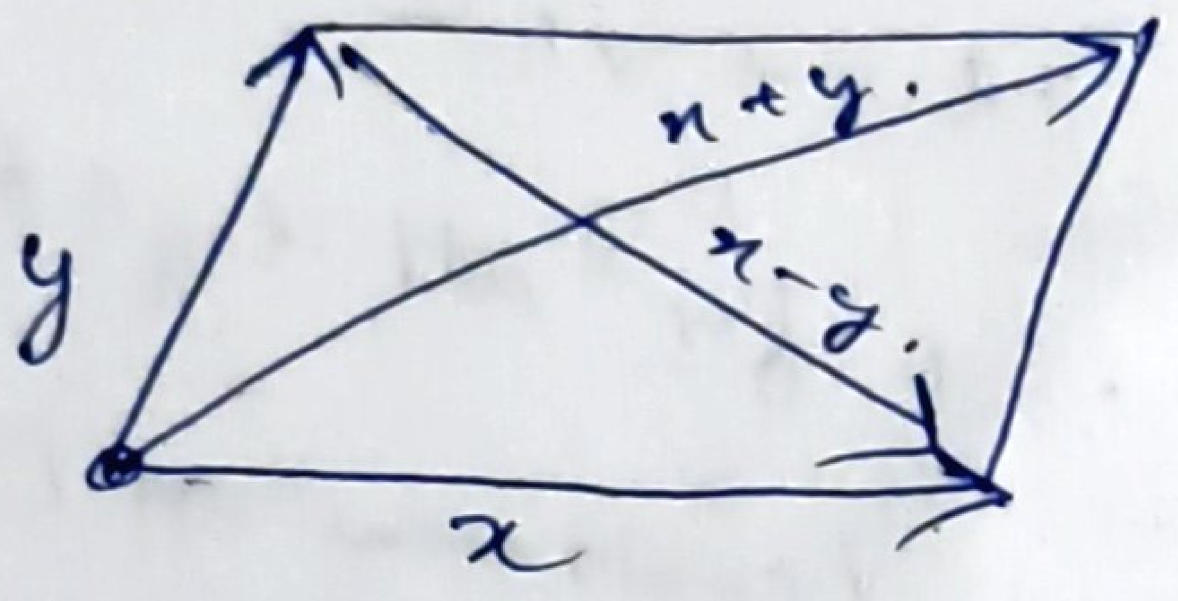
where $a = \|u\|^2$, $c = \|v\|^2$, $b = 2u \cdot v$.

$\boxed{(u \cdot v)^2 \leq \|u\|^2 \|v\|^2} \Rightarrow$ gives (1)

Parallelogram identity :

For any x, y in $(X, \langle \cdot , \cdot \rangle)$

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2).$$



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Theorem : All normed space satisfy parallelogram identity

But inner product need not satisfy parallelogram identity.

Sol^n. Take $\|\cdot\|_p$ - norm on $\mathbb{F}^n$ where $n \geq 2$ induce from an inner product

Take $x = (1, 1, 0, \ldots 0)$ and

$y = (1, -1, 0, \ldots 0)$ in $\mathbb{F}^n$.

where $\|x\|_p = \|(x_1, x_2 \ldots, x_n)\|_p$

$$= \left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p}$$

$$\|x\|_p = \left(\sum_{k=1}^{2} |x_k|^p\right)^{1/p}$$

$$= (1+1)^{1/p}$$

$$\|x\|_p = 2^{1/p}$$ 22

Similarly $\|y\|_p = 2^{1/p}$.

$$\|x+y\|_p = \|(1,1,0,\dots 0) + (1,-1,0 \dots 0)\|$$

$$= \|2,0,0,\dots 0\|$$

$$= 2.$$

Similarly. $\|x-y\|_p = 2$.

Note :- Here $\|\cdot\|_p$ is induced from inner product space

Now using parallelogram identity

$$\|x+y\|_p^2 + \|x-y\|_p^2 = 2^2+2^2 = 8$$

$$\Rightarrow 2\left(\|x\|_p^2 + \|y\|_p^2\right) = 2\left(2^{2/p} + 2^{2/p}\right)$$

$$= 2^1 \cdot 2^{2/p+1}$$

$$= 2^{2/p+2}$$

$$\Rightarrow 8 = 2^{2/p} + 2$$

if $p = 1, 3, 4, 5 \ldots$, then

$$\|x+y\|_p^2 + \|x-y\|_p^2 \neq 2\left(\|x\|_p^2 + \|y\|_p^2\right)$$

But if $p = 2$, then hold.

i.e parallelogram identity fail in inner product space $\|\cdot\|_p$ if $p \neq 2$.

<u>Another Example</u> :- $\left(C[0,1], \|\ \|_\infty\right)$

Take $f(x) = 1$ and $g(x) = x$.

$\|f\|_\infty = 1, \quad \|g\|_\infty = 1$ i.e $x \in [0,1]$

$$\|f+g\|_\infty = \|1 + \sup_{x\in[0,1]} x\|_\infty$$

$$= \sup_{x\in[0,1]} |1+1|$$

$$= 2$$

$$\|f-g\|_\infty = \sup_{x\in[0,1]} \|1\,\phi - x\|$$

$$= \sup_{x\in[0,1]} \|1-0\| = 1.$$

$$\|f+g\|_\infty = \sup_{x\in[0,1]} \|f+g\|$$

$$= \|1+1\| = 2.$$

Now used the parallelogram property, it will not satisfy

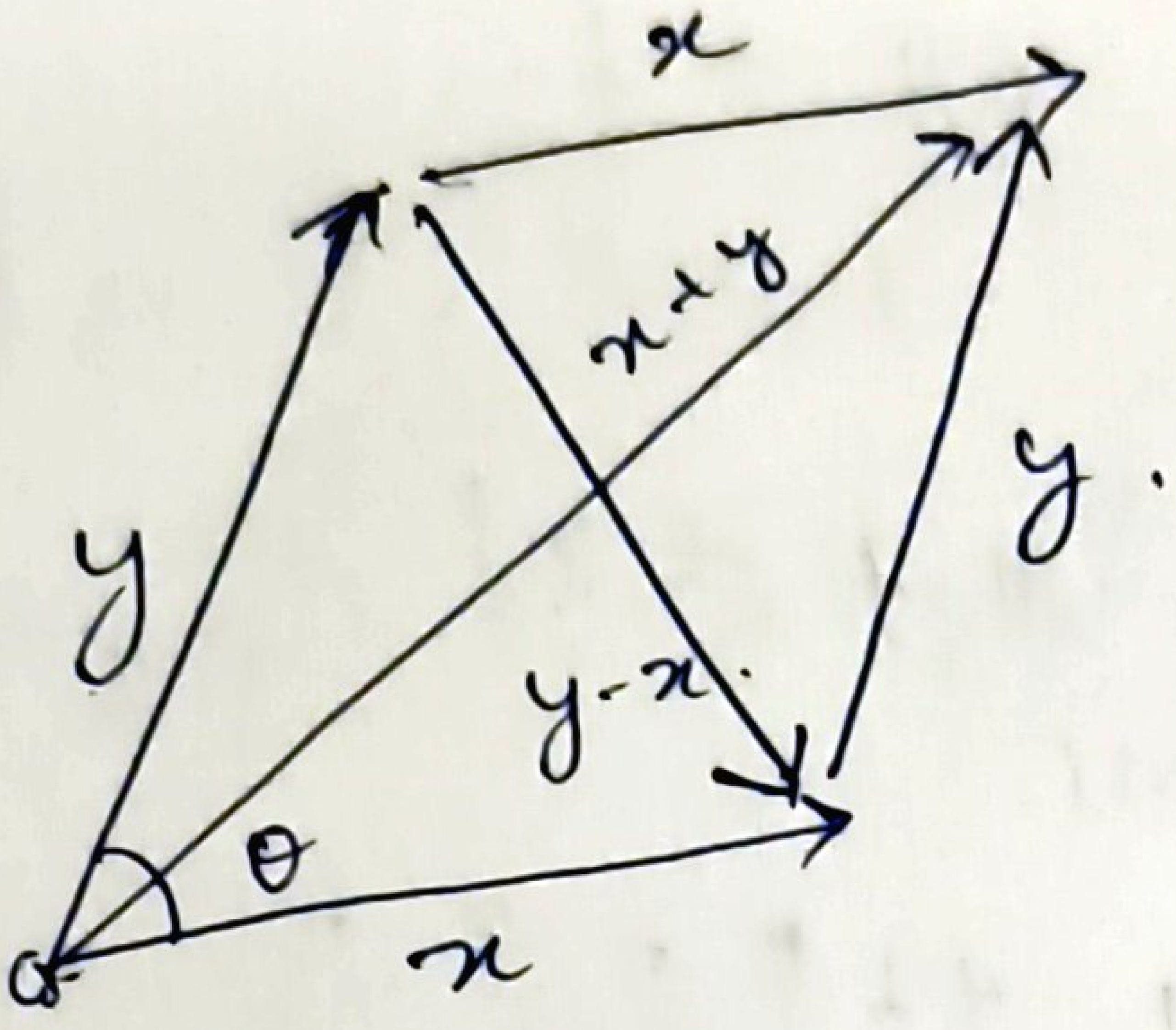
$$\|f+g\|_\infty^2 + \|f-g\|_\infty^2 = 5$$

$$\left(\|f\|_\infty^2 + \|g\|_\infty^2\right) = 4.$$

i.e $\|f+g\|_\infty^2 + \|f-g\|_\infty^2 \neq \left(\|f\|_\infty^2 + \|g\|_\infty^2\right)$

Not satisfy parallelogram identity

Construction of inner product from the norm on $\mathbb{R}^2$



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$$\langle x, y\rangle = \|x\|\,\|y\| \cos\theta$$

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\|x\|\,\|y\| \cos(\pi-\theta)$$

$$\|x-y\|^2 = \|x\|^2 + \|y\|^2 - 2\|x\|\,\|y\| \cos\theta$$

$$\|x+y\|^2 - \|x-y\|^2$$

$$= \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| \cos\theta$$

$$- \left(\|x\|^2 + \|y\|^2 - 2\|x\|\|y\|\cos\theta\right)$$

$$= 4\|x\|\|y\| \cos\theta \quad \left\{ \begin{array}{l} \cos(\pi-\theta) \\ = \cos(-\theta) = \cos\theta \end{array} \right\}$$

$$= 4\langle x, y\rangle$$

Polarization identities

Let V be inner product space over $\mathbb{F}$ (field) ($\mathbb{F}$ may $\mathbb{R}$ or $\mathbb{C}$), then the inner product space can be expressed as

$$\langle x, y \rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 \right)$$

(26)

i.e $4\langle x, y\rangle = \|x+y\|^2 - \|x-y\|^2$ if $\mathbb{F} = \mathbb{R}$

But if $\mathbb{F} = \mathbb{C}$, then

$$4\langle x, y\rangle = (\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2$$

General Rule in Complex inner product space.

$$\langle x, y\rangle = \frac{1}{4}\sum_{k=0}^{3} i^k \langle x+i^k y, x+i^k y\rangle$$

$$= \frac{1}{4}\sum_{k=0}^{3} i^k \|x+i^k y\|^2$$

$$= \frac{1}{4} \sum_{k=0}^{3} i^k \left(\|x\|^2 + \|y^2\| + \langle x, i^k y\rangle + \langle i^k y, x\rangle \right)$$

$$= \frac{1}{4} \sum_{k=0}^{3} \left(i^k \|x\|^2 + i^k \|y\|^2 + \langle x, y\rangle + i^{2k} \langle y, x\rangle \right)$$

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$$= \frac{4}{4} \langle x, y\rangle.$$

Here $\sum_{k=0}^{3} (i^k \|x\|^2 + i^k \|y\|^2.$

$$= \sum_{k=0}^{3} i^k \|x\|^2 + \|y\|^2$$

~~Here~~ $= \sum \quad i^0 \|x\|^2 + \|y\|^2 + i \|x\|^2 + \|y\|^2 + i^2 \|x\|^2 + \|y\|^2 + i^3 \|x\|^2 + \|y\|^2$

$$= 0$$

i,e $\sum_{K=0}^{3} (i^K \|x\|^2 + i^K \|y\|^2 = 0$

similary $\sum_{K=0}^{3} i^{2K} \langle y, x \rangle$

$= i^0 \langle y, x \rangle + i^2 \langle y, x \rangle + i^2 \langle y, x \rangle + i^3 \langle y, x \rangle$

$= \langle y, x \rangle + i \langle y, x \rangle - \langle y, x \rangle - i \langle y, x \rangle$

$= 0$

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$\sum_{K=0}^{3} i^{2K} \langle y, x \rangle = 0$

i,e $\boxed{\langle x, y \rangle = \frac{1}{4} \sum_{K=0}^{3} i^K \langle x + i^K y, x + i^K y \rangle}$

Polarization identity

HILBERT SPACE :-

$\Rightarrow$ A complete innerproduct space is called hilbert space. i.e every cauchy sequence with respect to the induced norm is convergent.

Note :- Norm is mean square metric ($\sqrt{\|\cdot\|}$ $= \langle x,x\rangle^{1/2}$)

or

A set H is called Hilbert space if it satisfy the given following properties

(1) H is vector space over $\mathbb{C}$ or ($\mathbb{R}^2$)

(2) H is equipped with an inner product $\langle \cdot , \cdot \rangle$ so that

(i) $f \longmapsto \langle f,g\rangle$ is linear on H for every fixed $g \in H$

(ii) $\langle f,g\rangle = \overline{\langle g,f\rangle}$

(iii) ~~f,g~~ $\langle f,f\rangle \geq 0$ for all $f \in H$.

where $\|f\| = \sqrt{\langle f,f\rangle}$

(3) $\|f\| = 0$ · if ~~and~~ and only if $f = 0$

(4) The cauchy - Schwarz and triangle inequalities hold $|(f,g)| \leq \|f\| \|g\|$

$\|f+g\| \leq \|f\| + \|g\|$ for $f, g \in H$

(5) H is complete in the metric

$d(f,g) = \|f-g\|$ (30)

(6) H is separable.

(7) $(H, \|\cdot\|)$ must satisfy the parallelogram identity.

Example :- Every hilbert space is a banach space. but every banach space need not be hilbert space.

~~proof~~ Take $(C[0,1], \|\ \|_\infty)$ is banach space

but not hilbert space

Because it doesnot satisfy ~~parallog~~

parallelogram identity.

(31)

Take $f(x) = 1$ and $g(x) = x$

$$\|x+y\|^2 + \|x-y\|^2 \neq 2(\|x\|^2 + \|y\|^2)$$

$$5 \neq 4.$$

\# ~~$l^p = \{ x = \{x$~~

\# $l^p = \{ x = \{x_n\}_{n \geq 1} : \|x\|_p^p = \sum_{k=1}^{\infty} |x_k|^p < \infty \}$

$\downarrow$

denote l^p space. for $1 \leq p < \infty$

$(l^p, \|\cdot\|_p)$ is a hilbert space if $p = 2$
but $p \neq 2$ then not hilbert

similarly $(l^p[a,b], \|\cdot\|_p)$ is a hilbert space if $p = 2$ but $p \neq 2$ then

... hilbert, ~~because~~

not hilbert space because if $p \neq 2$,

then $p = 1, 3, 4, 5, \ldots$

~~then part~~ parallelogram identity will

be fail.

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(*) if $(L^p[a,b], \|\cdot\|_p)$ then

$$\| f \|_p = \int_a^b |f(x)|^p dx.$$

proof: if $p \neq 2$, then $(L^p[a,b], \|\cdot\|_p)$ then

not hilbert space

Take $a = -1$, and $b = 1$.

$f(x) = 1 + x$, $g(x) = 1 - x$

$f, g \in L^p[-1, 1]$

$$\| f \|_p^p = \int_{-1}^{1} |f(x)|^p dx.$$

$$= \int_{-1}^{1} (1+x)^p dx = \frac{2^{p+1}}{p+1}.$$

$$\|g\|_p = \int_{-1}^{1} (1-x)^p dx = \frac{2^{p+1}}{p+1}.$$

$$\|f+g\|_p^p = \int_{-1}^{1} |(1+x) + (1-x)|^p dx$$

$$= \int_{-1}^{1} 2^p dx = 2^{p+1}$$

$$\|f-g\|_p^p = \int_{-1}^{1} |(1+x) - (1-x)|^p dx.$$

$$= 2^p \int_{-1}^{1} |x|^p dx.$$

$$= 2^p \cdot 2 \int_{0}^{1} |x|^p dx.$$

$$= 2^{p+1} \cdot \left. \frac{|x|^{p+1}}{p+1} \right]_0^1$$

$$= \frac{2^{p+1}}{p+1}$$

Now ~~HH~~ $\|f\|_p^p =$

parallelogram identity (using)

$$\|f+g\|_p^p + \|f-g\|_p^p = 2\left(\|f\|^2 + \|g\|^2\right)$$

$$\Rightarrow 2^{p+1} + \frac{2^{p+1}}{p+1} \neq 2\left(\frac{2^{p+1}}{p+1} + \frac{2^{p+1}}{p+1}.\right)$$

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$$\Rightarrow 2^{p+1} + \frac{2^{p+1}}{p+1} \neq 2\left(\frac{2 \cdot 2^{p+1}}{p+1}\right)$$

But if $p=2$, then parallelogram identity will satisfy.

$$\Rightarrow 2^{2+1} + \frac{2^{2+1}}{2+1} = 2\left(\frac{2 \cdot 2^{2+1}}{2+1}\right)$$

$$\Rightarrow 8 + \frac{8}{3} = \frac{2 \cdot 2 \cdot 8}{3}$$

$$\Rightarrow \frac{24+8}{3} = \frac{32}{3}$$

Pythagorean theorem :- If $x \perp y$ in an inner product space X, then we have.

$$\|x+\alpha y\| = \|x-\alpha y\| \text{ for all scalers } \alpha.$$

proof :- Let X be an inner product space, & and let $x, y \in X$. suppose that $x \perp y$, then $\langle x, y \rangle = 0$. So for every scalars α, we have

$$\|x+\alpha y\|^2 = \langle x+\alpha y, x+\alpha y \rangle$$

$$= \langle x, x \rangle + \bar{\alpha}\langle x, y \rangle + \alpha\overline{\langle x, y \rangle} + \alpha\bar{\alpha}\langle y, y \rangle$$

$$= \langle x, x \rangle + \bar{\alpha}\cdot 0 + \alpha\cdot\bar{0} + \alpha\bar{\alpha}\langle y, y \rangle$$

$$= \|x\|^2 + |\alpha|^2\|y\|^2$$

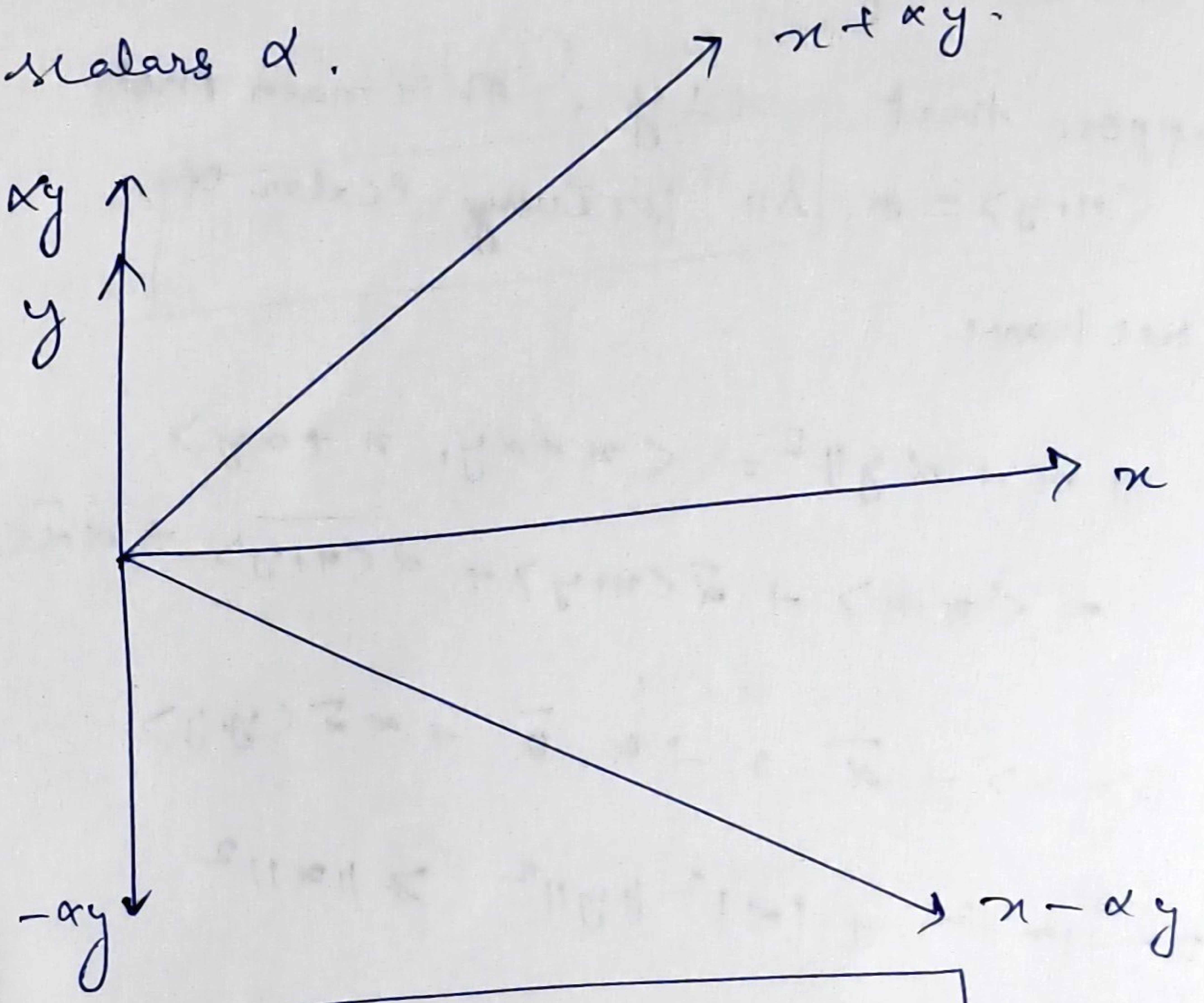
$$\|x-\alpha y\|^2 = \langle x-\alpha y, x-\alpha y \rangle$$

$$= \langle x, x \rangle - \bar{\alpha}\langle x, y \rangle - \alpha\overline{\langle x, y \rangle} + \alpha\bar{\alpha}\langle y, y \rangle$$

$= \langle x, x \rangle \pm \bar{\alpha} \cdot 0 - \alpha \cdot \bar{0} + \alpha \bar{\alpha} \langle y, y \rangle$

$= \|x\|^2 + |\alpha|^2 \|y\|^2$

$\therefore \|x + \alpha y\|^2 = \|x - \alpha y\|^2$ for every scalars α.



$$\boxed{\|x + \alpha y\| = \|x - \alpha y\|}$$

Theorem : In an inner product space, $x \perp y$ if and only if $\|x \pm \alpha y\| \geq \|x\|$ for all scalars α.

Sol. Let X be an inner product space and let $x, y \in X$.

Suppose that $x \perp y$. This mean that $\langle x, y \rangle = 0$. So for every scalar α, we have

$$\|x + \alpha y\|^2 = \langle x + \alpha y, x + \alpha y \rangle$$

$$= \langle x, x \rangle + \bar{\alpha} \langle x, y \rangle + \alpha \overline{\langle x, y \rangle} + \alpha \bar{\alpha} \langle y, y \rangle$$

$$= \langle x, x \rangle + \bar{\alpha} \cdot 0 + \alpha \cdot \bar{0} + \alpha \bar{\alpha} \langle y, y \rangle$$

$$= \|x\|^2 + |\alpha|^2 \|y\|^2 \geq \|x\|^2$$

$$\|x + \alpha\|^2 \geq \|x\|^2 \quad \text{—} (1)$$

Similarly, $\|x - \alpha y\|^2 = \langle x - \alpha y, x - \alpha y \rangle$

$$= \langle x, x \rangle - \bar{\alpha} \langle x, y \rangle - \alpha \overline{\langle x, y \rangle} + \alpha \bar{\alpha} \langle y, y \rangle$$

$= \|x\|^2 + |\alpha|^2 \|y\|^2$

$\geq \|x\|^2$

$$\boxed{\|x - \alpha y\|^2 \geq \|x\|^2} \quad \text{--- (2)}$$

from (1) and (2) we have

$$\boxed{\|x \pm \alpha y\| \geq \|x\|}$$

Theorem:- let V be the vector space of all continuous complex-valued functions on $I = [a,b]$.

let $X_1 = (V, \|\cdot\|_\infty)$, where

$$\|x\|_\infty = \max_{t \in [0,1]}$$

$$\|x\|_\infty = \max_{t \in [a,b]} |x(t)|$$

and let $X_2 = (V, \|\cdot\|_2)$, where

$$\|x\|_2 = \langle x, x \rangle^{1/2},$$

$$\langle x, y \rangle = \int_a^b x(t)\, \overline{y(t)}\, dt$$

Then the identity mapping $x \longrightarrow x$ of (X_1 onto X_2) is uniformly continuous.

$$\left(X_1 \xrightarrow{\text{onto}} X_2 \right)$$

proof:- let d denote the metric induced by the inner product on X_2. Then for any $x, y \in X_2$, we have

$$|d(x,y)|^2 = \langle x-y, x-y \rangle$$

$$d(x,y) = \langle x-y, x-y \rangle^{1/2}$$

$$|d(x,y)|^2 = \int_a^b (x(t) - y(t)) \overline{(x(t) - y(t))}\, dt$$

$$= \int_a^b |x(t) - y(t)|^2 dt$$

$$\leq \int_a^b \max_{r \in [a,b]} |x(r) - y(r)|^2 dt.$$

$$= \max_{r \in [a,b]} |x(r) - y(r)|^2 \int_a^b dt.$$

$$= \|x - y\|_\infty^2 (b-a)$$

$$\|x - y\|_\infty \sqrt{b-a}.$$

So $d(x,y) \leq \|x-y\|_\infty \sqrt{b-a}$ for all $x, y \in X_2$

let's choose a real number $\varepsilon > 0$, and let δ be any real number such that

$$0 < \delta \leq \frac{\varepsilon}{\sqrt{b-a}}.$$

Then for any element $x, y \in X_1$ for which

$\|x - y\|_\infty < \delta.$

we have

$$d(x, y) \leq \|x - y\|_\infty \sqrt{b-a}$$

$$< \delta \sqrt{b-a} \leq \frac{\varepsilon}{\sqrt{b-a}} \times \sqrt{b-a}$$

$$\leq \varepsilon = \varepsilon.$$

$d(x, y) < \varepsilon.$

this show that the identity mapping

$$x \longmapsto x \text{ of } X_1 \xrightarrow{\text{onto}} X_2$$

is uniformly continuous.

Total Orthonormal Sets and Sequences :-

① (**Bessel inequality**) : let (e_k) be an orthonormal ~~basis~~ sequence in an inner product space X.

For every $x \in X$, we have

$$\sum_{k=1}^{\infty} |\langle e_k, x\rangle|^2 \le \|x\|^2$$

Soln. ~~let~~ let $y = \sum_{k=1}^{n} \langle e_k, x\rangle e_k$.

$$\|y\|^2 = \langle y, y\rangle = \left\langle \sum_{k=1}^{\infty} \langle e_k, x\rangle e_k, \sum_{k=1}^{\infty} \langle e_k, x\rangle e_k \right\rangle$$

$$= \sum_{k=1}^{n} \langle e_k, x\rangle^2 \|e_k\|^2$$

$$= \sum_{k=1}^{n} \langle e_k, x\rangle^2$$

$$= \sum_{k=1}^{n} |\langle e_k, x\rangle|^2$$

For $x \in X$, we have $z = x - y$.

and $\langle x, y \rangle = \sum_{k=1}^{n} \langle e_k, x \rangle \langle x, e_k \rangle$

$= \|y\|^2$. ——— (i)

Now for each $x \in X$, we have.

$0 \leq \|x - y\|^2 = \langle x - y, x - y \rangle$

$= \|x\|^2 - 2\langle x, y \rangle + \|y\|^2$

$= \cancel{\|x\|^2 - 2\langle x, \sum_{k=1}^{\infty} \langle e_k, x \rangle e_k \rangle + \|y\|^2}$

Since $y =$

$= \|x\|^2 - 2\langle x, \sum_{k=1}^{\infty} \langle e_k, x \rangle e_k \rangle + \|y\|^2$

Since $y = \sum_{k=1}^{n} \langle e_k, x \rangle e_k$. $+ \|y\|^2$

$= \|x\|^2 - 2\langle x, y \rangle + \|y\|^2$

From (I), we have,

$$= \|x\|^2 - 2\|y\|^2 + \|y\|^2$$

$$= \|x\|^2 - \|y\|^2$$

~~i.e~~ $\cancel{0 \le \|z\|}$

$$0 \le \|x\|^2 - \|y\|^2$$

Therefore $\|y\|^2 \le \|x\|^2$

implies $\|y\| \le \|x\|$

i.e $\cancel{\sum_{k=1}^{n} \langle e_k, x \rangle^2 \le \|x\|^2}$

$$\sum_{k=1}^{m} |\langle e_k, x \rangle|^2 \le \|x\|^2$$

Hence proved.

Fourier coefficient and Fourier series with orthonormal sequence.

⇒ If $\{e_n\}$ is an orthonormal sequence in a Hilbert space H, then for every $x \in H$,

→ (x, e_n) is the n-th fourier coefficient of x with respect to $\{e_n\}$

→ $\sum_{n=1}^{\infty} (x, e_n) e_n$ is the fourier series with respect to $\{e_n\}$

Total orthonormal sets :-

let X be an innerproduct space and let M be a subset. Then M is called total if $\overline{span(M)} = X$

Example :- Take $X = \{ f : [0,1] \to [0, \infty) \text{ is continuous} \}$.

norm, $\|f\|_1 = \int_0^1 |f(x)|\, dx$.

$M = \{ f_n \in X : f_n(x) = x^n \}$.

$Span(M)$ = the set of all polynomial

we know that set of all polynomial function is continuous, and dense in $\mathbb{R}$ / X

$\therefore \boxed{\overline{span(M)} = X}$

$span(M) = \{a_0 + a_1 x + a_2 + a_3 x^2 + \dots + a_n x^n\}$

$M = \{f_n \in X : f_n(x) = x^n\}.$

Theorem : Let M be a subset of an inner product space X. Then

(1) If M is total in X, then $x \perp M$ implies $x = 0$

(2) Suppose that X is complete. ~~If~~ ~~$x \perp$~~ If ~~x~~ $x \perp M$ implies $x = 0$, then M is total in X.

proof :- Suppose $x \perp M$. Given that M is total then there is a sequence (x_n) in span (M) such that $x_n \longrightarrow x$ and continuity of inner product, $\langle x_n, x \rangle = \langle x, x \rangle = 0$

$$\Rightarrow x = 0.$$

Note : Separable space if it contains a countable dense subsets.

Theorem (separable Hilbert space):

Let H be a Hilbert space. Then

(1) If H is separable, every orthonormal set in H is countable.

(2) If H contains an orthonormal sequence which Total in H, then H is separable.

proof: (1) Take $V \subseteq H$ is an orthonormal set, and take x, y such that $x \neq y$ and $x, y \in V$.

Now ~~HO~~ $\| x - y \| = \sqrt{\langle (x-y), (x-y) \rangle}$

$$= \sqrt{x^2 + y^2}$$

$$= \sqrt{2} \quad \text{put } x = 1, y = 1.$$

i.e ~~If~~ $d(x,y) = \| x - y \| = \sqrt{2}$.

induces discrete metric

so if V is uncountable, a set dense in H is also uncountable, We know that in ~~in~~ discrete metric space, the whole space is the only dense set i,e every space is a dense subsets of itself.

Now if H is a separable space, then V is also ~~metric~~ separable metric space.

Here V is the only subsets of itself that is dense in V since V has discrete metric/topology

so V must be <u>separable</u>.

(2) If $\{e_k\}$ is an total orthonormal set, then the set D, consisting of all linear combination $\lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_n e_n$

where $n \in \mathbb{N}$ and

$$\lambda_k = a_k + i b_k \text{ with } a_k \in \mathbb{Q},\ b_k \in \mathbb{Q}$$

for $k = 1, 2, \ldots n$.

~~λ_k~~ ~~Then~~ So the set D is a countable ~~dur~~ dense in H because λ_k in the form $\mathbb{Q} + i\mathbb{Q}$ countable

$\Rightarrow \mathbb{Q}$ is ~~countable~~.

Example : Take $H = L^2[0, 2\pi]$

~~$V = \{f_n\}$~~

$$V = \left\{ f_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx} ; n \in \mathbb{N} \right\}$$

$\mathbb{N}$ is countable.

* Every hilbert space need not be separable space.

Example: Take $H = L^2(\mathbb{R})$

and $V = \left\{ f_n(x) = \begin{cases} 1, & \text{if } x = n \\ 0 & x \neq n \end{cases} \right.$

where $n \in \mathbb{N} \Big\}$.

Here image from in $\{0,1\}^{\mathbb{N}}$

we know that $\{0,1\}^{\mathbb{N}}$ is uncountable.

so it is not separable space.

Riesz Representation Theorem :

let H be a hilbert space. For each $g \in H$

let $f_g : H \longrightarrow \mathbb{R}$ be defined for all $h \in H$ by

$$f_g(h) = \langle h, g \rangle$$

where ~~fg~~ f_g is a continuous linear functional.

Because $f_g(h_1 + h_2) = \langle h_1 + h_2, g \rangle$

$$= \langle h_1, g \rangle + \langle h_2, g \rangle$$

$$= f_g(h_1) + f_g(h_2)$$

Now take scalar $\lambda \in \mathbb{R}$, we have

$$f_g(\lambda h) = \langle \lambda h, g \rangle = \lambda \langle h, g \rangle$$

$$= \lambda f_g(h)$$

i.e $\boxed{f_g(\lambda h) = \lambda f_g(h)}$