Homotopy :-

Let X, Y be topological space and $f, g : X \longrightarrow Y$

continious maps.

A homotopy from f to g is a continious function $F : X \times [0,1] \longrightarrow Y$ satisfying

$$F(x,0) = f(x)$$
$$F(x,1) = g(x) \text{ for all } x \in X.$$

~~If such~~ a homotopy exist,

A homotopy between two function f and g from a space X to a space Y is a continuous map G from

$$G : X \times [0,1] \longrightarrow Y \text{ defined by}$$

$$G(x,0) = f(x)$$

$$G(x,1) = g(x)$$

Homotopy is a path in the mapping space $Map(X,Y)$ from the first function to the second.

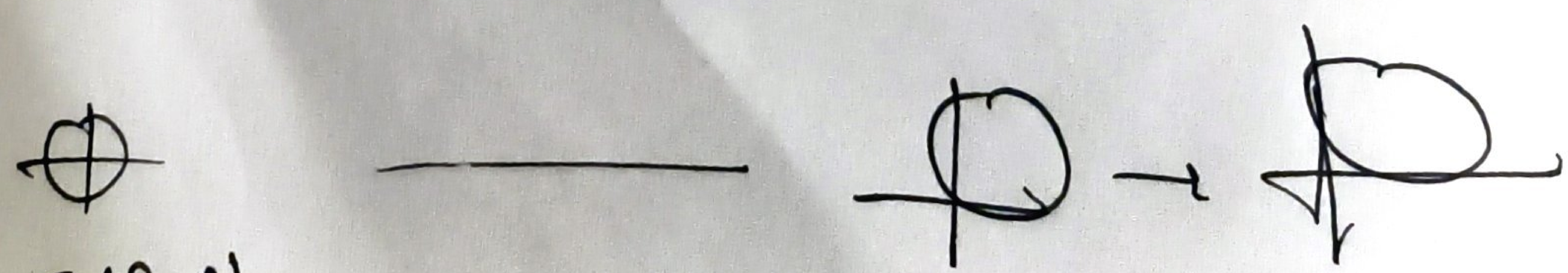
Ex. The map

$$F : S_1 \times [0,1] \longrightarrow \mathbb{R}^2$$

$$F(\theta,0) = (\cos\theta, \sin\theta) = f$$

$$F(\theta,1) = (\cos\theta + 1, \sin\theta + 1) = g$$

is a homotopy between the maps $f : S^1 \rightarrow \mathbb{R}^2$ and $g : S^1 \rightarrow \mathbb{R}^2$ defined as.



$F(\theta,0)$ $F(\theta,1)$

Retract :-

let X be a topological space and let $A \subset X$ be a topological ~~space~~ subspace.
Then A is said ~~be~~ to be a retract of X if there exist a continuous function $f: X \longrightarrow A$ such that

B. called a retraction map

$$f \circ i = id_A$$

where i is inclusion map

$$f \circ i = I_a$$

i.e $i : A \longrightarrow X$

$i(a) = a$ where $a \in A$.

$$f \circ i(a) = f(a) = a \quad \text{(identity map)}$$

$$f: X \longrightarrow A$$

A subspace A of X is called a retract of X if there is a continuous map $f: X \longrightarrow \cancel{A}$ (called a retraction) such that for all $x \in X$ and all $a \in A$

1. $f(x) \in A$
2. $f(a) = a$.

Example: let D^2 be the closed unit disk in $\mathbb{R}^2$.

$$D^2 = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 1\}.$$

and let $\frac{1}{2}D^2$ be the closed disk centre at the origin with radius $\frac{1}{2}$ in $\mathbb{R}^2$, i.e

$$\frac{1}{2}D^2 = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq \frac{1}{4}\}$$

Then $\frac{1}{2}D^2$ is a retraction of D^2.

Define a retraction ~~map~~ continuous map

$$f : X \longrightarrow A \text{ by}$$

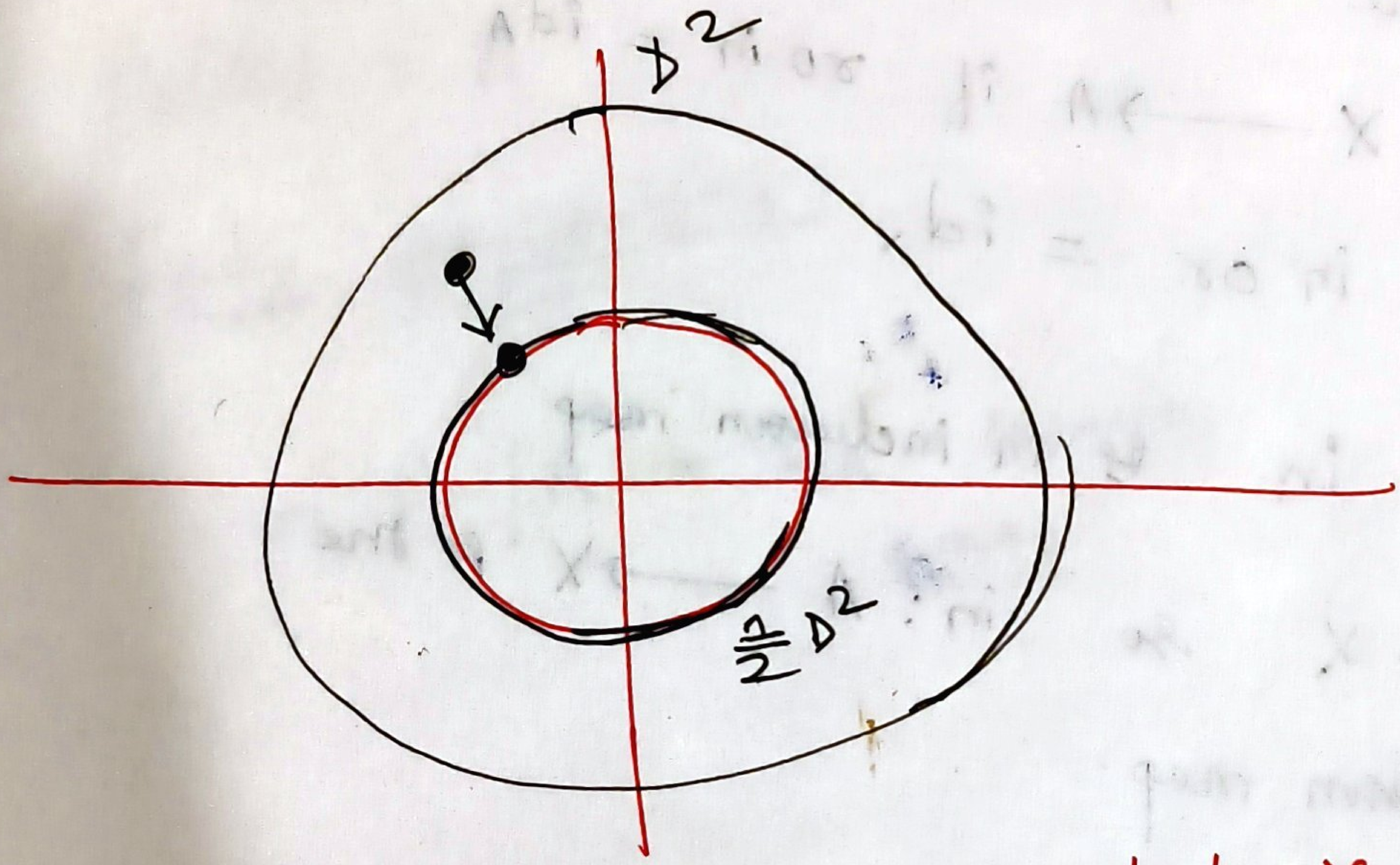
$$f(x,y) = (a,b)$$

where ~~(a,b)~~ $d((x,y),(a,b))$

$$= \inf_{(c,d) \in \frac{1}{2}D^2} d((x,y),(c,d))$$

Each point $(x,y) \in D^2$ is mapped to the point (a,b) in $\frac{1}{2}D^2$ whose distance from (x,y) is minimized.

Therefore $\frac{1}{2}D^2$ is a ~~rtc~~ retract of D^2.



Point in the closed unit disk are mapped to the closest point in the closed disk centre at the ~~orid~~ origin with radius

4.

Deformation Retract subspaces:

- let X be a topological space and let $A \subset X$ be a topological subspace. Then A is said to be a Deformation Retract of X. if there exist a continuous function

$$r : X \longrightarrow A \text{ if } r \circ in = id_A \text{ and}$$

$$in \circ r = id_X$$

where in is an inclusion map

Since $A \subset X$ so $in : A \longrightarrow X$ is the inclusion map.

A subspace A of X is called a deformation retract of X if there is a homotopy

$F : X \times I \longrightarrow X$ such that for all $x \in X$ and $a \in A$

(1) $F(x, 0) = x$

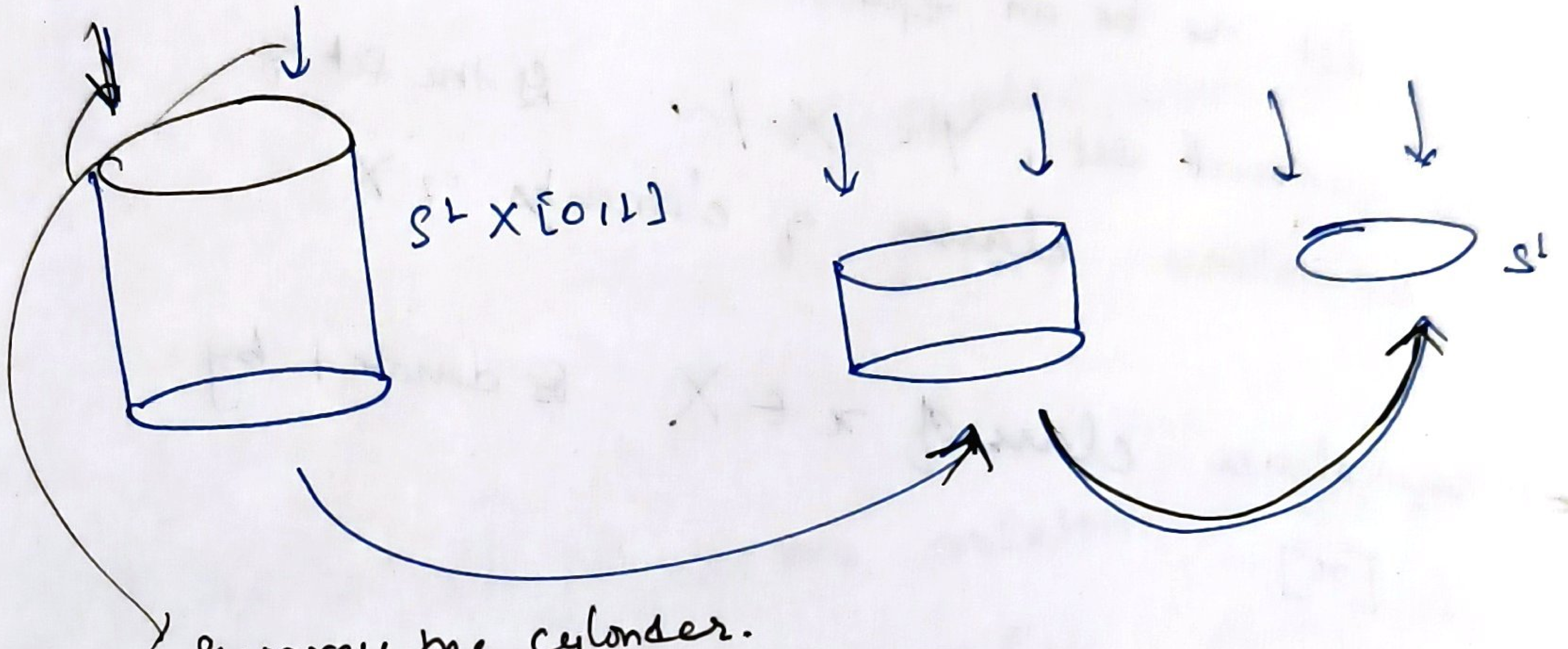
(2) $F(x, 1) \in A$

(3) $F(a, 1) = a$.

Example : $A = S^1$ which is the unit circle

and $X = S^1 \times [0,1]$ which is the unit cylinder.

Then A is a deformation retract of X.



suppress the cylinder.

Example (2) : Consider the space $A = \{p\}$

and $X = \{p, q\}$ where $p, q \in \mathbb{R}^2$ and $p \neq q$

and equip X with the discrete topology.

Then A is not a deformation retract of X

Since there doesn't exist a continuous function

from $\gamma : X \longrightarrow A$ such that

$i_n \circ \gamma = id_X$ and

$\gamma \circ i_n = id_A$

Since $X = \{p, q\}$ is not connected.

$p \cap q = \phi$

but A is connected.

Strong Deformation retract :

A subspace A of X is called a strong deformation retract of X if there is a homotopy $F : X \times I \longrightarrow X$ (called a retract) such that for all $x \in X$, $a \in A$, and $t \in I$, we have

(1) $F(x,0) = x$

~~(2)~~ $\cancel{F(x,1) \in F(x,1)}$

(2) $F(x,1) \in A$, and

(3) $F(a,t) = a$

If the last equation is required only for $t = 1$, the retract is called simply a deformation retract.

A deformation retraction of a space X onto a subspace A is a family of maps

$f_t : X \longrightarrow X$, $t \in I$, such that

$f_0 = 1_X$

$f_1(X) = A$

$f_t|_A = id_A$ for all t.

Difference between deformation retraction and retraction:

① For any $x_0 \in X$, $\{x_0\} \subset X$ has a retract. Choose $r : X \longrightarrow$ ~~to~~ $\{x_0\}$ to be the unique map to the one-point set. Then certainly $r(x_0) = x_0$.

However, $\{x_0\} \subset X$ ~~has~~ only has a deformation retraction if X is contractible.

If $\{x_0\}$ is a deformation retract of X

then there has to be a family of maps

$f_t : X \longrightarrow X$ such that $f_0(x) = x$

$f_1(x) = x_0$

and $f_t(x_0) = x_0$ for every t.

i.e $F : X \times I \longrightarrow X$ such that

for all $x \in X$ and $x_0 \in \{x_0\}$

$F(x,0) = x$

$F(x,1) \in \{x_0\}$

$F(x_0, 1) = x_0$

This gives a homotopy from id_X to the constant map at x_0, which makes X contractible.

let $A \subseteq X$ a subspace. A is a deformation retract of X if the identity map id_X is homotopic to a map sending all of X into A such that all points of A are fixed during the homotopy

i,e, there exist a continuous

$$H : X \times I \longrightarrow X \text{ s.t}$$

$$H(x,0) = x$$

$$H(a,t) = a \quad \forall\, a \in A$$

$$H(x,1) \in A$$

such a homotopy H is called a deformation retraction of X onto A.

The function $r : X \longrightarrow A$ defined $r(x) = H(x,1)$ is a retraction of X onto A.

Example : If $X = \mathbb{R}^n \setminus \{0\}$, then

$H : X \times I \longrightarrow X$ defined by

$$H(x,t) = \frac{tx}{\|x\|} + (1-t)x$$ is

a deformation retract into S^{n-1}.

If $a \in S^{n-1}$, then $H(a,t) = \frac{ta}{1} + (1-t)a$

$= a$

Q. Construct an explicit deformation retraction of $\mathbb{R}^n - \{0\}$ onto S^{n-1}.

Sol. Let $X = \mathbb{R}^n \setminus \{0\}$. And $i : S^{n-1} \longrightarrow X$ an inclusion map.

$r : X \longrightarrow S^{n-1}$ defined by

$r = \frac{x}{\|x\|}$ is a retraction.

Now, define $H : X \times I \longrightarrow X$ by

$H(x,t) = (1-t)x + t\frac{x}{\|x\|}$ for

$0 \leq t \leq 1$.

we see that H is a homotopy between the identity map of X and the retraction of X onto S^{n-1}.

And it is clear that $i \cdot r = 1_X$

and $r \cdot i = 1_{S^{n-1}}$.

Hence H is the deformation retraction of X onto S^{n-1}.

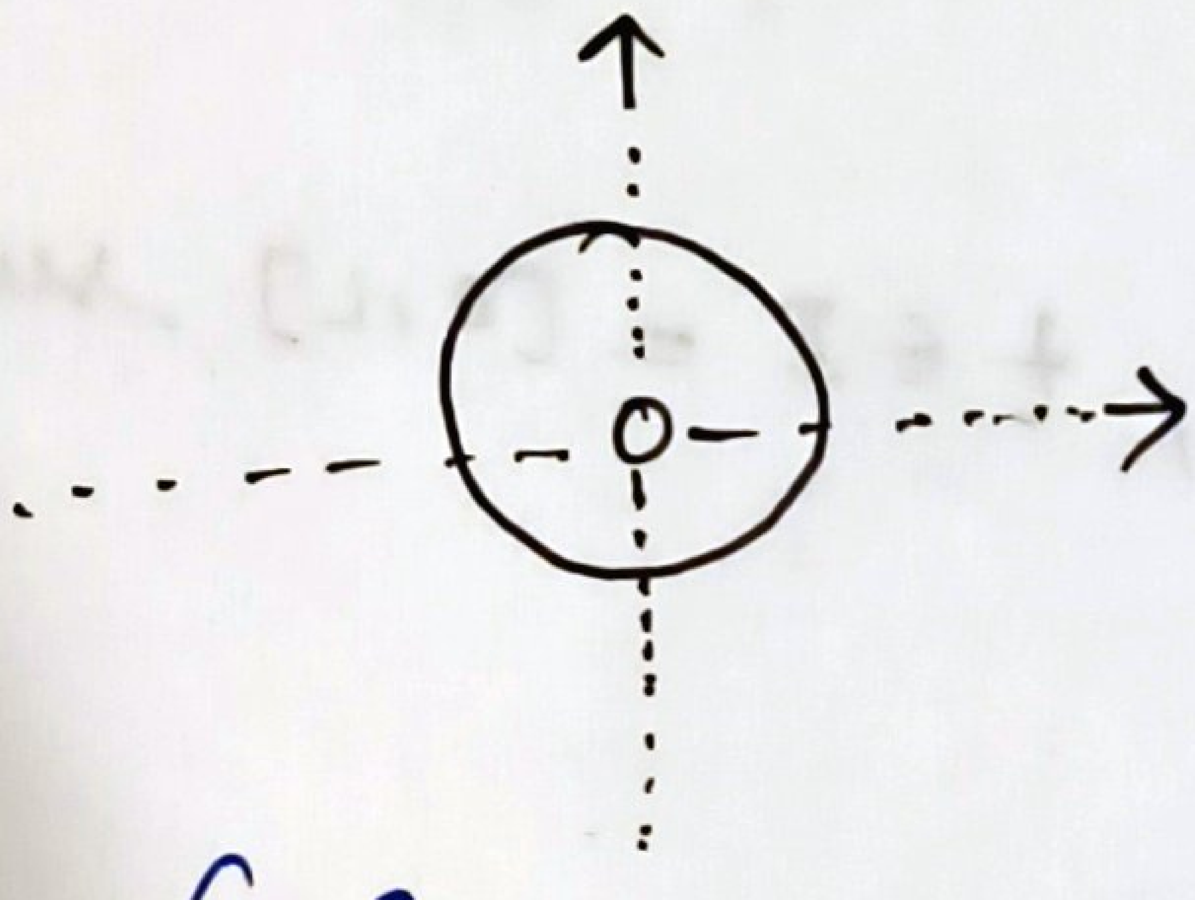
A is a deformation retraction of X if the id_X (identity) map is homotopic to a retraction of X onto A.

Retraction :-

A pair of spaces (X, A) consist of a topological space X and a subspace $A \subset X$.

If $A = \{x\}$, then we write (X, x) and call this is a pointed space.

Consider the pair of spaces $(\mathbb{R}^2 - \{0\}, S^1)$ or the pair $(S^1, (1,0))$



$(\mathbb{R}^2 - \{0\}, S^1)$

$(S^1, (1,0))$

A subset $A \subset X$ is a retract of X if there is a map $r : X \longrightarrow A$ (the retraction) such that the restriction satisfies

$$r|_A = id_A$$

i.e $r(a) = a$ for $a \in A$

Example: The set $\mathbb{R}^2 - \{0\}$ retract to S^1 via $r(x) = \frac{x}{\|x\|}$

Deformation retract :-

Let (X, A) be a pair of spaces.

X deformation retracts to A i.e

A is called a deformation retract of X

if there exists a one-parameter family of function

$$f_t : \cancel{X} \longrightarrow X, \quad t \in I = [0,1] \text{ such that}$$

$$f_0 = Id_X$$

$$f_1(X) = A$$

$$f_t|_A = id_A, \quad t \in [0,1]$$

and the map $X \times I \longrightarrow X$
$(x, t) \longrightarrow f_t(x)$ is continuous.

Example :- ① $\mathbb{R}^n$ deformation retract to 0

by means of $f_t(x) = (1-t)x$.

$f_0(x) = x$ (id_x)

$f_1(x) = \{0\}$

$F : X \times I \longrightarrow X$

$(x, t) \longrightarrow f_t(x)$

is continuous.

i.e $F(x,t) = f_t(x)$

Q. show that $\mathbb{R}^n - \{0\}$ deformation retract to S^{n-1} via

$$f_t(x) = (1-t)x + \frac{tx}{\|x\|}$$

Sol.

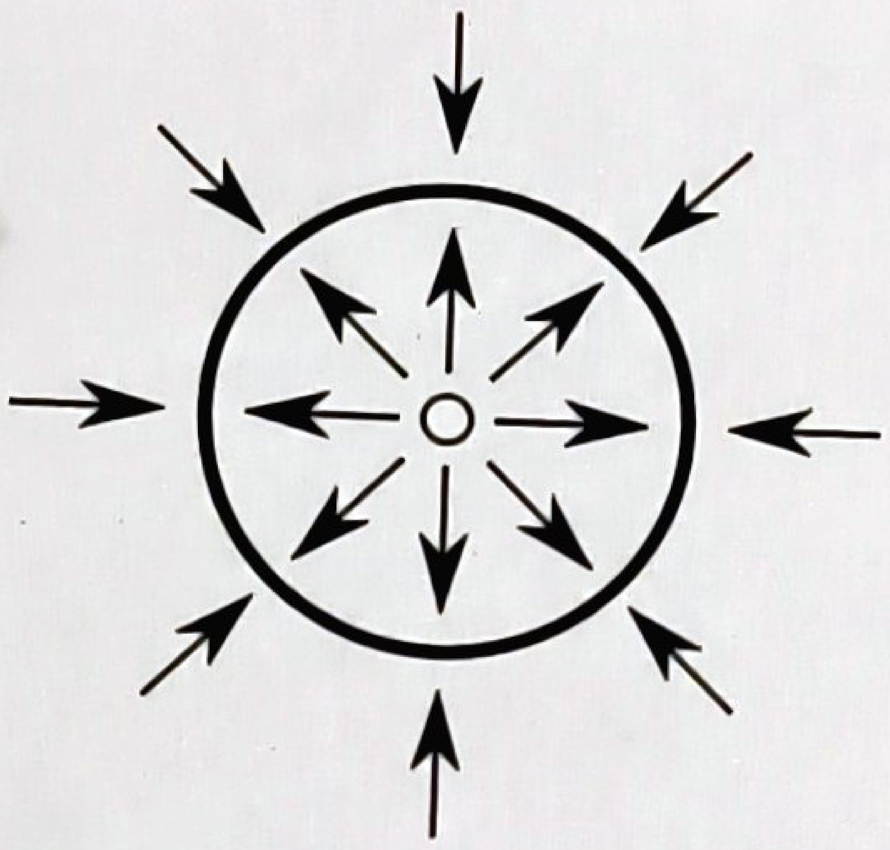


Figure 3.2: Deformation of a punctured plane $\mathbb{R}^2 - \{0\}$ onto the circle S^1.

Set of all points in ~~2-D~~ plane $\mathbb{R}^2$, except for the origin.

Consider the collection of maps

$$f_t : \mathbb{R}^n - \{0\} \longrightarrow \mathbb{R}^n - \{0\}$$

$$x \longrightarrow f_t(x) = (1-t)x + t\frac{x}{\|x\|}$$

Then f_t defines a deformation retract of ~~$\mathbb{R}^n$~~ $\mathbb{R}^n - \{0\}$ ~~and~~ onto S^{n-1}.

(1) The map $F(x,t) = f_t(x)$ is clearly continuous.

(2) At time $t=0$, we have the identity

$$f_0(x) = (1-0)x + 0 \cdot \frac{x}{\|x\|} = x$$

(3) At time $t=1$, the image is contained in S^{n-1}

$$\|f_1(x)\| = \left\| \frac{x}{\|x\|} \right\| = 1.$$

(4) all maps for S^{n-1}

$$y \in S^{n-1} \Rightarrow y = \frac{y}{\|y\|}$$

$$\Rightarrow f_t(y) = y$$

Q. Prove that the subset $S^1 \times \{x_0\}$ is a retract but not a deformation retract of $S^1 \times S^1$.

Sol. Suppose x_0 is any point of S^1

~~let~~ let us define the map

$$r : S^1 \times S^1 \longrightarrow S^1 \times \{x_0\} \text{ by}$$

$$r(x, y) = (x, x_0) \text{ where } x, y \in S^1.$$

This map is continuous and fixes pointwise the subspace $S^1 \times \{x_0\}$

Let $i : S^1 \times x_0 \longrightarrow S^1 \times S^1$ be the inclusion map and

$$r \circ i(x, x_0) = r(i(x, x_0))$$
$$= r(x, x_0)$$
$$= (x, x_0)$$

Hence, r is a retraction from $S^1 \times S^1$ to $S^1 \times \{x_0\}$

Deformation retractions are homotopy equivalences, and induce isomorphisms of fundamental groups.

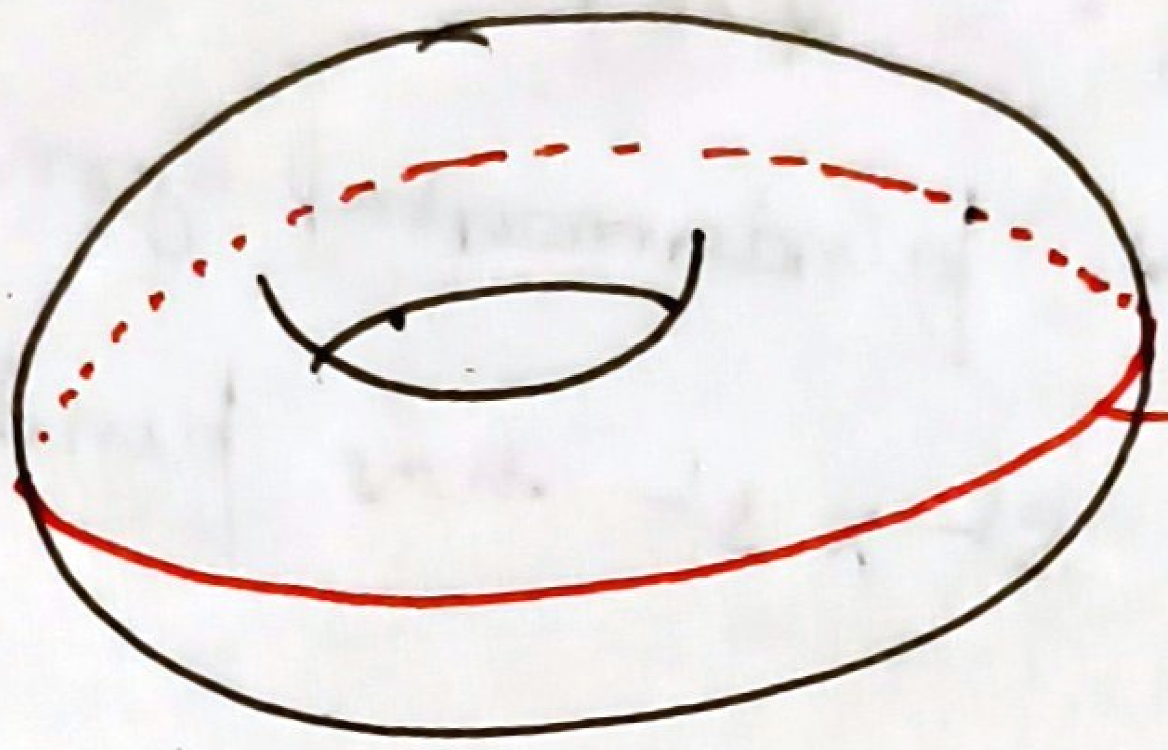
However $S^1 \times \{x_0\}$ is isomorphic to S^1 which has fundamental group $\mathbb{Z}$, whereas $S^1 \times S^1$ has fundamental group $\mathbb{Z} \times \mathbb{Z}$.

$\mathbb{Z}$ is cyclic but $\mathbb{Z} \times \mathbb{Z}$ is not cyclic.

So these two groups are not isomorphic,

So there can be no deformation retract of $S^1 \times S^1$ onto $S^1 \times \{x_0\}$.

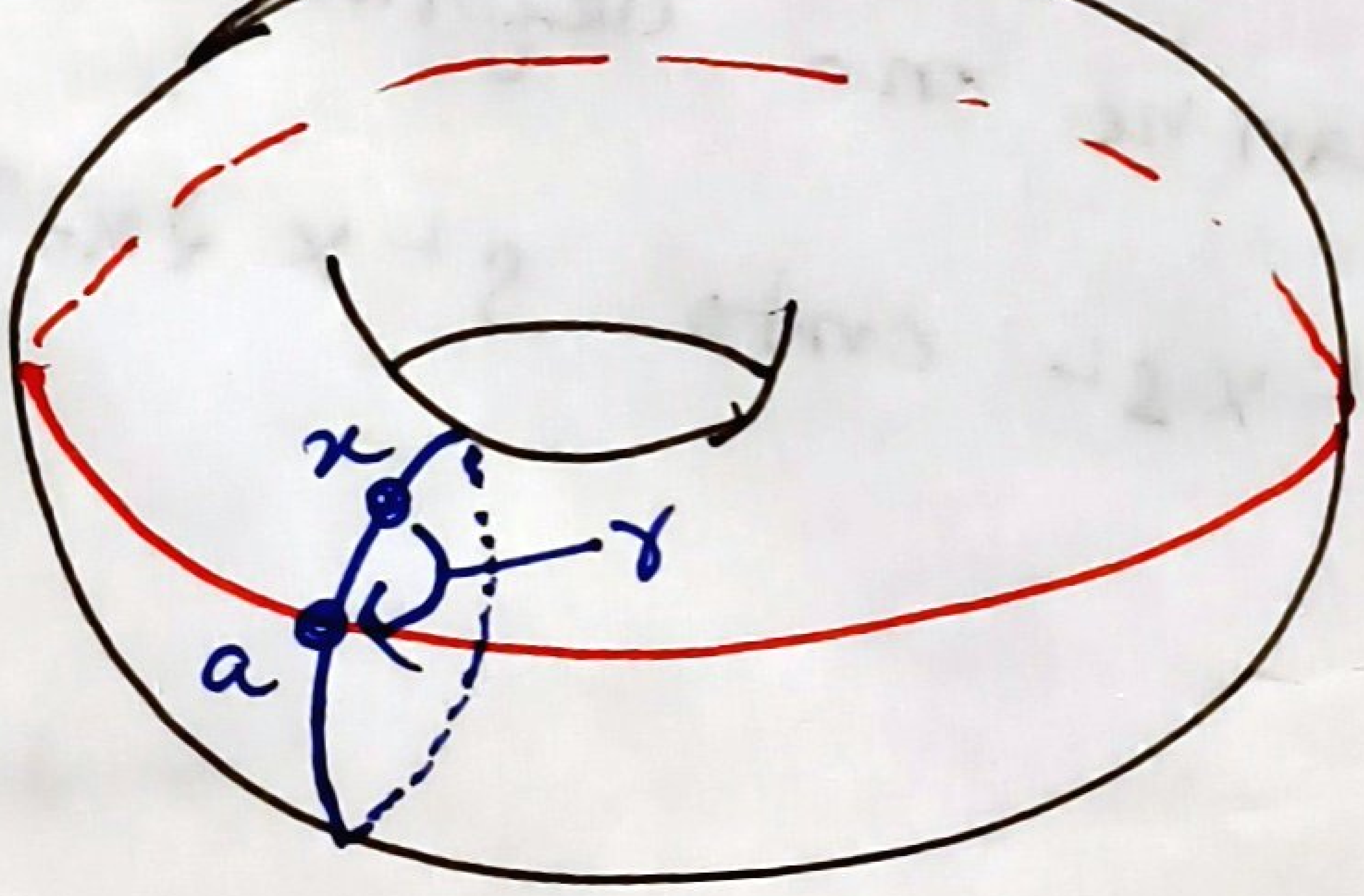
$\longrightarrow S^1 \times S^1 = X$



$A = S^1$ or

$S^1 \times \{x_0\} \cong S^1$

Also A is called equatorial circle.



Showing that A is a retract of X

Define a function $r : X \longrightarrow A$ as follows

~~cut~~ 26

cut $X = S^1 \times S^1$ upto vertical ~~section~~ section

Then each circle intersect A uniquely

Define $\gamma(x)$ to be the unique point $a \in A$ for which the unique circle intersect both x and a.

It is easy to see that γ is a continuous map. This is because of

U is open subset of X, then $\gamma^{-1}(U)$ will be union of open interval on the equatorial circle. So $\gamma^{-1}(U)$ is open.

Furthermore, we have that $\gamma(a) = a$ for all $a \in A$, and

so $\gamma \circ in = id_A$

So A is a retract of X.

Example of retraction but not deformation retraction.

~~Consider the sphere $S^1 = \{v \in \mathbb{R}^2 \mid \|v\| = 1\}$~~

Take $X = S^1$, $A = \{x\}$ where a point $x \in S^1$.

Since singleton is a retract in any space

so $r(x) = x$

$\Rightarrow \{x\}$ is ~~rato~~ retract of S^1

But $\{x\}$ is not a deformation retract since S^1 is not contractible.

Contractable :

A space X is contractable iff for any space T any two continuous maps $f, g : T \longrightarrow X$ are homotopic.

If Y is contractable then any continuous map $f : X \longrightarrow Y$ is null-homotopic.

proof:- If Y is contractable then ~~there~~ there is a homotopy $H : Y \times I \longrightarrow Y$ such that $H(x,0) = id_Y$ and ~~$H(x,$~~ ~~$H(x, x_0$~~ $H(x,1) = x_0 = c(y)$ where $x_0 \in Y$ and $c(y)$ denote the constant map $c : Y \longrightarrow x_0$

Since f is continuous, so we take

$G(x,t) = H(f(x), t)$.

Then G is continuous and ~~$G(x,t)$~~.

$$G(x,0) = H(f(x),0) = f$$

$$G(x,1) = H(f(x),1)$$

$$= x_0 = c(y)$$

$\Rightarrow f$ is null-homotopic.

i.e f is homotopic to a constant map.

Theorem :

Let X be a topological space and $f: X \longrightarrow S^n$ a continuous map from X to n-dimensional sphere which is not onto & i.e $f(X)$ is a ~~prope~~ proper subset of S^n. Then ~~Reso~~ f is null-homotopic.

Soln. Assume that there is $y_0 \in S^n$ such that $y_0 \notin \mathrm{im}(f)$. It is well known that there is a homeomorphism

$$\phi : S^n \setminus \{y_0\} \longrightarrow \mathbb{R}^n \text{ by}$$

Stereographic projection.

Then we have an induced map

$$\phi \circ f : X \longrightarrow \mathbb{R}^n$$

Since $\mathbb{R}^n$ is contractible, then there is $c \in \mathbb{R}^n$ such that $\phi \circ f$ is homotopic to the constant map c.

Take any homotopy

$H : I \times X \longrightarrow \mathbb{R}^n$ form

$\phi \circ f$ to c.

$H(x, 0) = id_{\mathbb{R}^n} = H((\phi \circ f(x)), 0)$

$= f(x)$

$H(x, 1) = c$

loop f is nullhomotopic $\Rightarrow \pi(S^2)$ is trivial.

Base point :-

The common starting and ending point x_0 is called base point.

The set of all homotopy classes $[f]$ of loops $f : I \longrightarrow X$ at the base point x_0 is denoted by $\pi_1(X, x_0)$

i.e ~~$\pi_1(x$~~ $\pi_1(X, x_0)$ is the set of all homotopy classes $[f]$ of loop f with the base point $x_0 = f(0)$

Since for a path $f : I \longrightarrow X$, f is a loop if $\boxed{f(0) = f(1)}$.

\# A homotopy class is an equivalence class under homotopy.

$\pi_1(X, x_0)$ is a group with respect to the product $[f][g] = [f \cdot g]$

Since it satisfy three properties.

① Associative

② identity : $[c]$ where c is the constant loop i.e $c(s) = x_0$ for any s.

③ Inverse . The inverse of $[f]$ will be $[\bar{f}]$. , where $\boxed{\bar{f}(s) = f(1-s)}$

proof :-

Theorem :- If X is a path connected space and x_0, x_1 are two distincts points of X, then the group $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are isomorphics

proof :- let X be a path connected space and $x_0, x_1 \in X$.

By the definition of path connectedness there exist some path

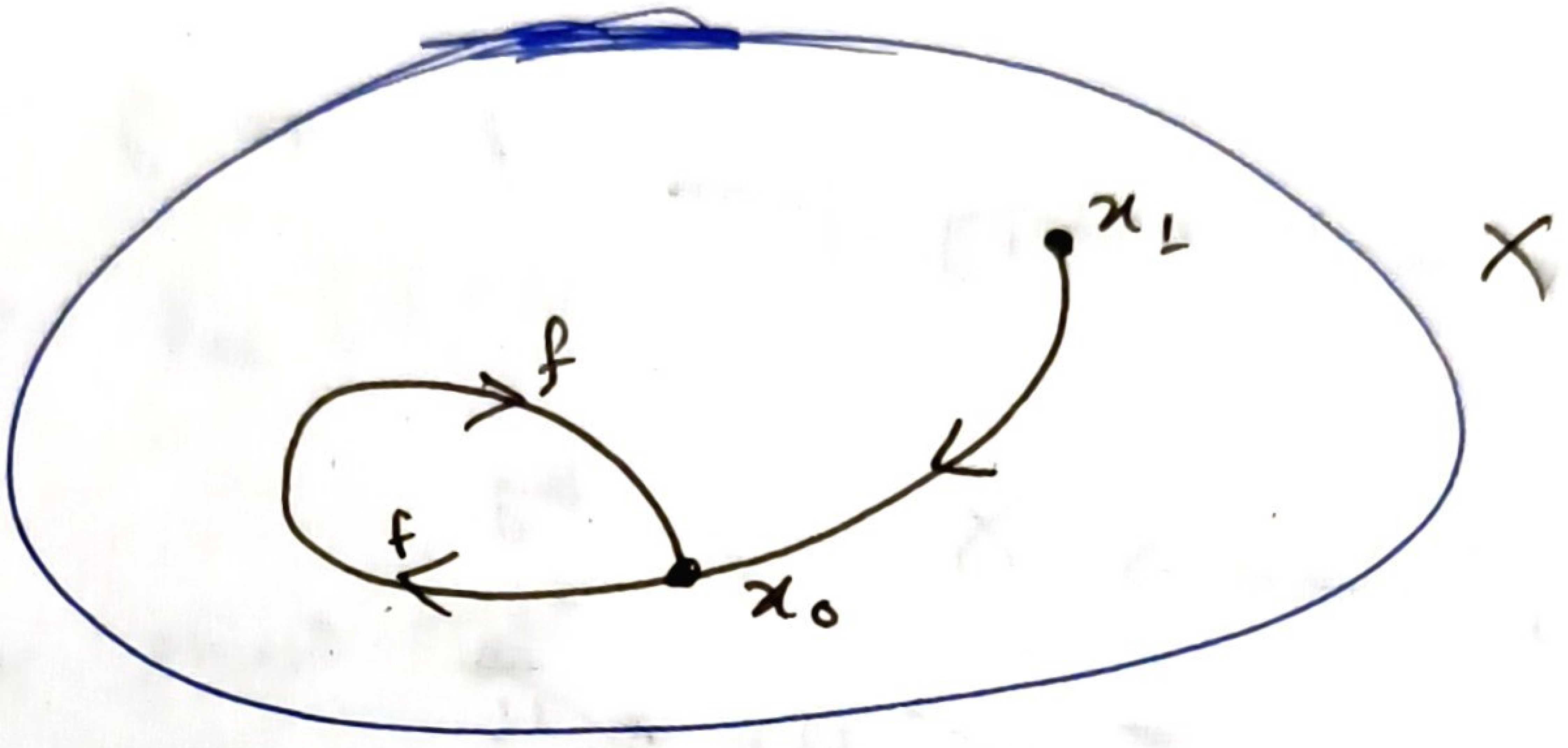
$$u : I \longrightarrow X.$$ in X

from x_0 to x_1 with the inverse path $\bar{u} : I \longrightarrow X$ defined by

$$\bar{u}(t) = u(1-t)$$ from x_1 to x_0

as shown in fig.

N~~ow we to~~



Now we define a map

$$\beta_u : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1)$$

$$\beta_u([f]) = [\bar{u} * f * u]$$

$$\forall \ [f] \in \pi_1(X, x_0)$$

where β_u is known as change of base point map.

step ① :- To prove β_u is well defined.

~~let $g \in [f]$~~, then $f \simeq g$

let $[g] \in \pi_1(X, x_0)$ and

$[f] \in \pi_1(X, x_0)$

Such that $[f] = [g]$.

Let H be a homotopy from f to g

and let

$F : I \times I \longrightarrow X$ by

$$F(s,t) = \bar{u} * H_t * u$$

$$F(0,t) = u * f * u$$

$$F(s,t) = \bar{u} * H_t * u$$

$$F(0,t) = \bar{u} * f * u$$

$$F(1,t) = \bar{u} * g * u$$

Since H is continuous by pasting lemme.

i.e there from f, u are continuous

$\Rightarrow$ F is continuous

$[\bar{u} * f * u] = [\bar{u} * g * u]$

$\Rightarrow \beta_u([f]) = \beta_u([g])$

thus show that β_u is well defined

and β_u is homomorphism

Now To prove: β_u is homomorphism.

proof:- Suppose any $[f], [g] \in \pi_1(X, x)$,

then $\beta_u([f] \circ [g])$

$= \beta_u([f] * [g])$

$= [\bar{u} * (f * g) * u]$

$= [\bar{u} * f * u \; * \bar{u} * g * u]$

$\Rightarrow$

Since $u * \bar{u} \simeq e_x$ rel $\dot{I}$

mean that $u * \bar{u}$ and e_x are homotopic relative to the subspace $\dot{I} = \{0, 1\}$ of I

$= [(\bar{u} * f * u) * (\bar{u} * g * u)]$

$= [\bar{u} * f * u] \circ [\bar{u} * g * u]$

$= \beta_u [f] \circ \beta_u [g]$

i.e $\beta_u ([f] \circ [g]) = \beta_u [f] \circ \beta_u [g]$

$\forall \; [f], [g] \in \pi_1 (X, x_0)$

To show β_u is an isomorphism :-

Take β_u inverse $= \beta_{\bar{u}}$

$\beta_{\bar{u}} : \pi_1 (X, x_1) \longrightarrow \pi_1 (X, x_0)$ by

$\beta_{\bar{u}} [f] = [\bar{\bar{u}} * f * \bar{u}]$

$\Rightarrow B_{\bar{u}}[f] = [u * f * \bar{u}]$

where $\boxed{\bar{\bar{u}} = u}$

then $(B_{\bar{u}} \circ B_u)([f]) = B_{\bar{u}}(B_u([f]))$

$= B_{\bar{u}}([\bar{u} * f * u]$

since $B_u([f]) = [\bar{u} * f * u]$

$= [\bar{\bar{u}} * \bar{u} * f * u * \bar{u}]$

$= [u \times \bar{u} * f * u * \bar{u}]$

$= [f]$

since $u * \bar{u} \simeq e_x$ rel $\dot{I}$

$\Rightarrow (B_{\bar{u}} \circ B_u)[f] = B_{\bar{u}}(B_u[f])$

$= [f]$

Similarly $\beta_u * \beta_{\bar{u}}$ is identity homomorphism

i.e $\beta_u(\beta_{\bar{u}}[f]) = [f]$

Therefore $\beta_{\bar{u}} = \beta_u$

$\beta_{\bar{u}}^{-1} = \beta_u$

This show that β_u is an isomorphism of group

i.e isomorphism from $\pi_1(X, x)$ to $\pi_1(X, y)$

Homotopy equivalent :-

① Homotopy :-

Two continuous map $f, g : X \longrightarrow Y$ are said to be homotopic if there is a continuous map $F : X \times I \longrightarrow Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$

The map F is called a homotopy ~~betw~~ between f ~~and~~ and g

i.e $f \simeq g$

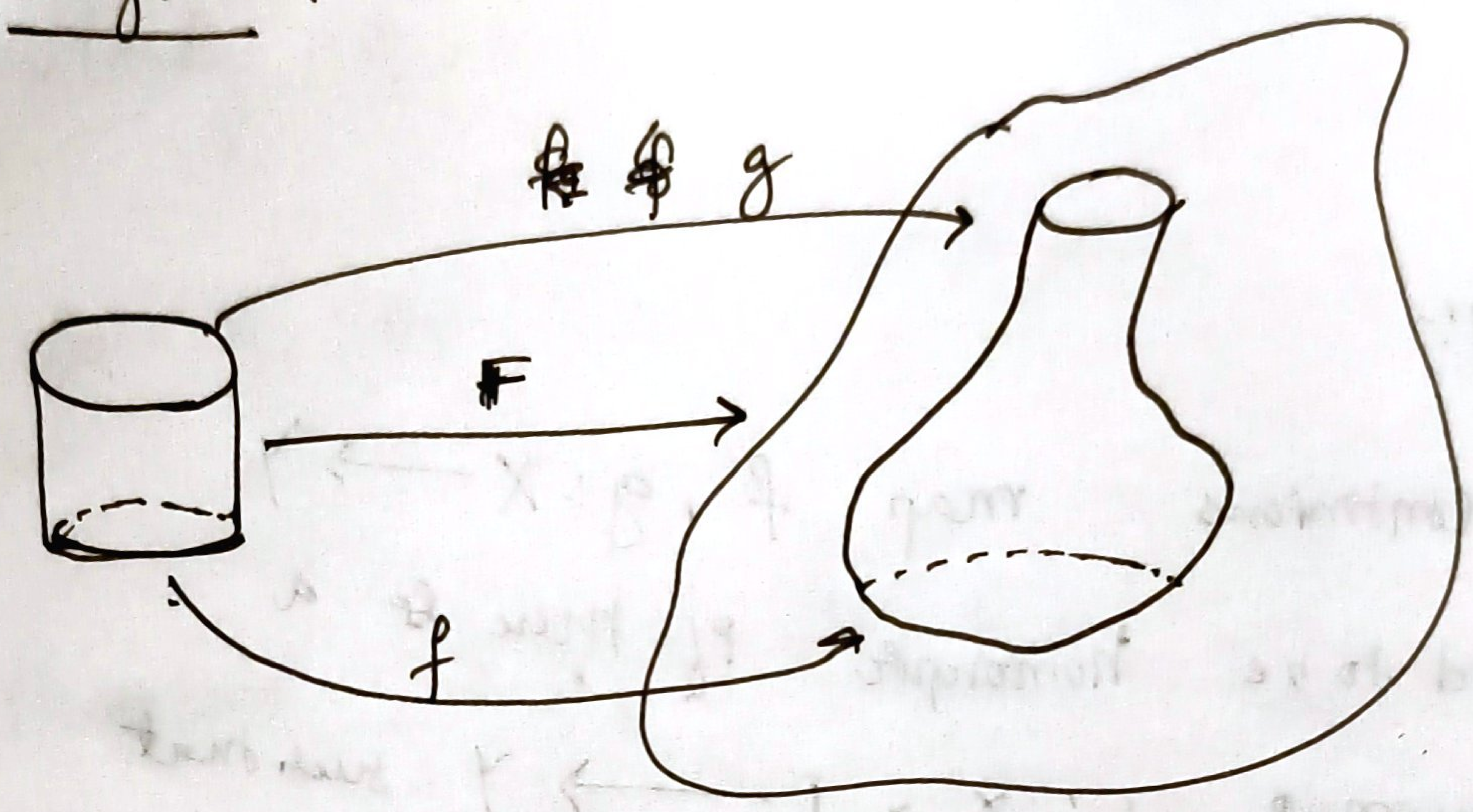
$F : f \simeq g$

For each $t \in [0, 1]$, we denote $F(x, t)$ by

~~for~~ $f_t(x)$

$F(x, t) = f_t(x) : X \longrightarrow Y$ is a continuous map

Diagram.



Homotopy equivalence / (Homotopy equivalent) :-

Two space X and Y are of the same homotopy type if there exist

Continuous map

$$f : X \longrightarrow Y$$

$$g : Y \longrightarrow X$$

such that

$$g \circ f = gf \simeq 1_X : X \longrightarrow X$$

$$f \circ g = fg \simeq 1_Y : Y \longrightarrow Y$$

~~$gf \simeq g \circ f$~~

$gf = g \circ f \simeq id_X$ (identity function)

$fg \approx f \circ g \simeq id_Y = I_Y$

Then the map f and g are called homotopy equivalences.

$\rightarrow$ X and Y are homotopy equivalent.

Contractible spaces :-

A space X is said to be contractible if it is homotopy equivalent to a ~~point~~ point or (singleton)

i.e ~~Suppose~~ if there is an x_0 in X such ~~that~~ that the identity mapping $id_X : X \longrightarrow X$ is homotopic to a constant mapping $c_{x_0} : X \longrightarrow X$

Here $id_X : X \longrightarrow X$ is defined by

$id_X(x) = x$ for $x \in X$

and $c : X \longrightarrow X$ is defined by

$c(x) = x_0$ for $x \in X$ also $x_0 \in X$.

~~then a homotopy~~ $\cancel{F : id_X \to}$

then a homotopy $F : id_X \simeq c$ is called

a contraction of the space X to the point x_0.

or **Another definition of contractible space :-**

Let X be a topological space and there is

a point $x_0 \in X$

If there is a continuous map $F : X \times [0,1] \longrightarrow X$ and

such that $F(x,0) = f(x) = x$

$F(x,1) = g(x) = c(x) = x_0$ for all

$x \in X$, then X is known

as contractible

i.e $f(x) = x$ mean $id_X(x)$

$g(x) = c(x) = x_0$ (constant map)

Theorem:- Any convex subspace X of $\mathbb{R}^n$ is Contractible

proof:- let $X \subset \mathbb{R}^n$ be convex set.

and let $x_0 \in X$.

let $g : X \longrightarrow X$ be the constant map ~~g(x)~~ $g(x) = c(x) = x_0$

Note: $g(x) = c(x)$ $\forall x$

Then any continuous map

$F : X \times [0,1] \longrightarrow X$ defined by

$F(x,t) = f_t(x) = (1-t)x + t x_0$

where $x, x_0 \in X$ and $t \in [0,1]$.

Now $F(x,0) = x = f(x)$

$F(x,1) = x_0 = c(x) = g(x)$

Since X is convex set, so F take values in X ~~(x,t) ∈ X~~

i.e $F(x,t) = f_t(x) \in X$

This implies that there is a homotopy between the $f(x)$ and $g(x)$

Hence $F: f(x) \sim g(x)$

$\Rightarrow X$ is contractible

Therefore H is a contraction of X to the point $x_0 \in X$.

Theorem :- If X is contractible space and $x_0 \in X$, then $\pi_1(X, x_0) = 0$

proof Given that X is contractible space this implies that X is an convex subsets of $\mathbb{R}^n$.

let x_0 be a base point in X.

let $f: I \longrightarrow X$ be a loop base at x_0.

Since X is contractible space so

$f \sim x_0 \text{ rel } \dot{I}$

i,e there is a homotopy between

f and x_0

$\Rightarrow [f] = [x_0]$ — (1)

Now let $c : I \longrightarrow X$ be a continuous curve such that $c(0) = c(1) = x_0$

Then $F : I \times X \longrightarrow X$ is defined by

by

$\cancel{F(x,t) = (1-t)f}$

$$F(x,t) = (1-t)\,c(x) + t x_0$$

$F(x,0) = c(x)$ where $x \in [0,1]$

$F(x,1) = x_0$ for $x \in [0,1]$

F is continuous by pasting lemma

this implies $c(x)$ is homotopic to x_0

$\Rightarrow c(x)$ is homotopic to the constant curve

From (I) we have

$$[f] = [x_0]$$

This implies that $[c(x)] = [f] = [x_0]$

$$\Rightarrow [c] = [f] = [x_0]$$

$[f] \in \pi_1(X, x_0)$ since the set of all homotopy classes $[f]$ of loops $f : I \longrightarrow X$ at the base point x_0 is denoted by $\pi_1(X, x_0)$

This implies that

$$\pi_1(X, x_0) = \{[x_0]\}$$

$$\pi_1(X, x_0) = \{0\}. \text{ is trivial.}$$

Some times we denoted by $\{1\}$

i.e $\pi_1(X, x_0) = \{1\}$.

Theorem :- If X is a simply connected space, then any two path in X having the same initial ~~p~~ and final points are homotopic.

proof :-

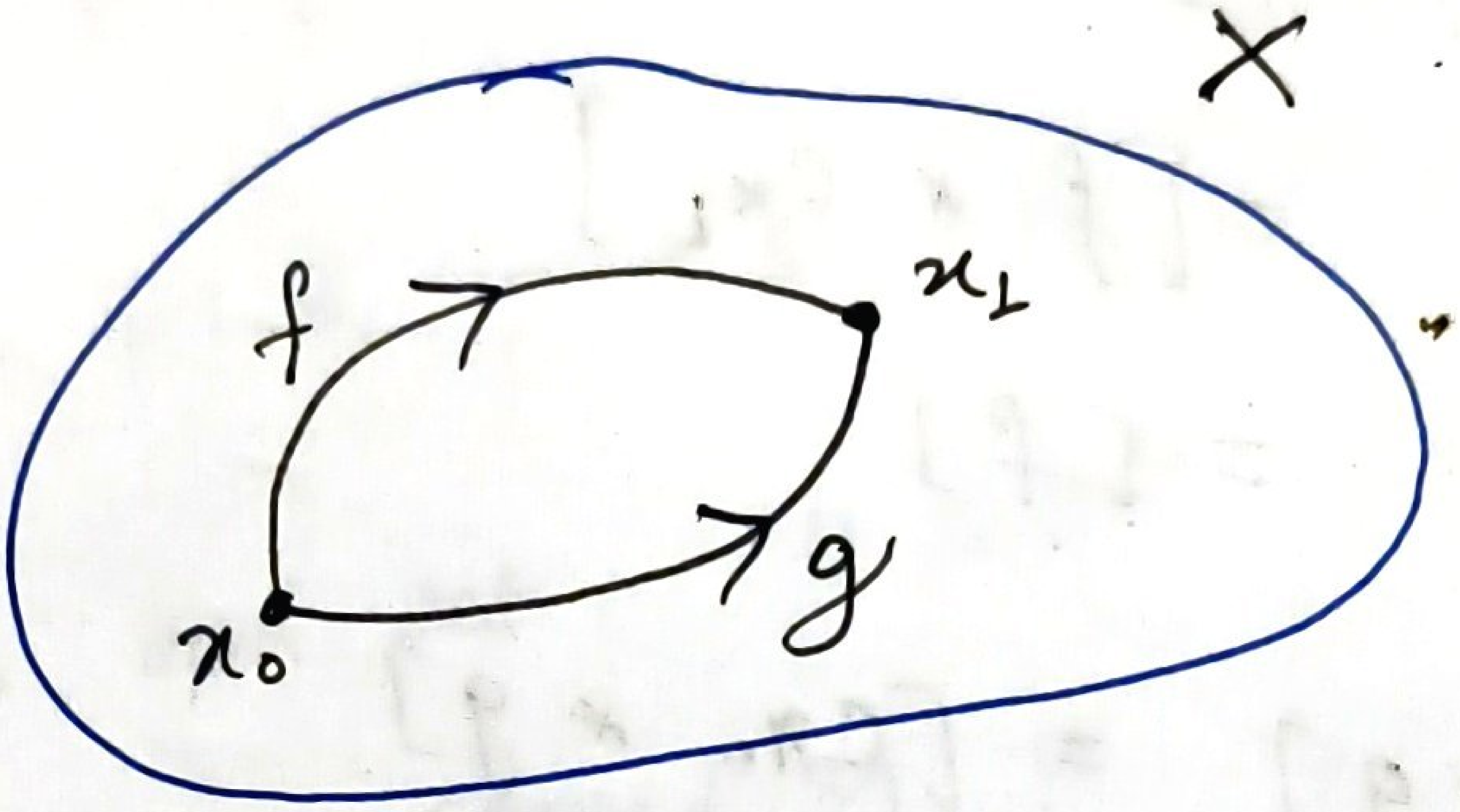
Note :- A topological space X is called simply connected if it is path connected for all $x \in$ and $\pi_1(X, x_0) = 0$ for some $x_0 \in X$.

~~p~~ Suppose that X is simply connected and let $x_0, x_1 \in X$.

Since X is path connected, x_0 and x_1 are connected by a path.

let f ~~&~~ g be two paths connecting ~~x_0 and x_1~~ x_0 and x_1

Let f and g be ~~too~~ two path in X from x_0 to x_1. as shown in given below figure.



We know that X is simply connected

$$\Rightarrow \pi(X, x_0) = 0$$

Therefore $f * \bar{g} = f * g^{-1}$ is a loop in X based at x_0

and $f * \bar{g} \simeq c_{x_0}$

$$\Rightarrow f * \bar{g} * g \simeq c_{x_0} * g$$

$$f * (\bar{g} * g) \simeq c_{x_0} * g$$

$$f * c_{x_1} \simeq c_{x_0} * g$$

$$f \simeq g$$

$[f] = [g]$.

Since $[(f * \bar{g}) * g] = [f * (\bar{g} * g)]$

$= [f * c_{x_1}]$

$= [f]$

and $[(f * \bar{g}) * g] = [c_{x_0} * g]$

$= [g]$

Therefore f and g are path homotopics,

i.e ~~there~~ if there is only one homotopy class of path connecting a base point x_0 to itself, then all loops at x_0 are homotopics to the constant loop and $\pi_1(X, x_0) = 0$

Covering space :-

Let $p : \mathbb{R} \longrightarrow S^1$ be continuous surjective map. An open set $U \subseteq S^1$ is said to be evenly covered ~~at~~ by p if

① $p^{-1}(U) = \bigcup V_\alpha$ is a disjoint union of open set in $\mathbb{R}$

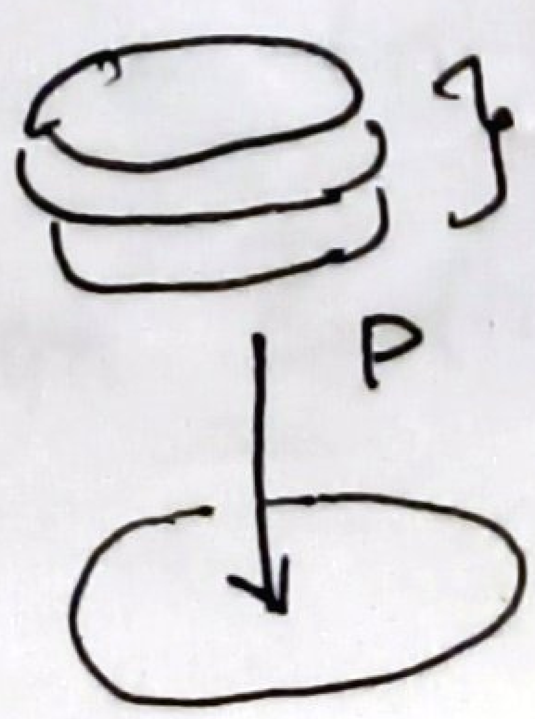
or

$= \bigcup V_\alpha$ is a disjoint union of open interval in $\mathbb{R}$

② $p|_{V_\alpha} : V_\alpha \longrightarrow U$ is a homeomorphism

Simple meaning :-

$p^{-1}(U) = \bigcup V_\alpha$ is a collection of sheet/slice



$p^{-1}(U) = \bigcup V_\alpha$

open set V_α are called sheets

~~For~~ If every point $b \in S^1$ has a neighborhood U that is evenly covered by p, then p is a covering map and $\mathbb{R}$ is a covering space of S^1.

S^1 is called the base of the covering.

~~for any point~~ $\cancel{x = e(x_1, x_2)}$ $x \in (x_1, x_2)$

For any point $b = (x, y) \in S^1$ such that $x > 0$, the set

$\cancel{U = \{(x_1, x_2)}$

$U = \{(x, y) \in S^1 \mid x > 0\}$ is an open nbd of ~~see~~ b

$$p^{-1}(U) = \bigcup_{n \in \mathbb{Z}} \left(n - \tfrac{1}{4}, n + \tfrac{1}{4}\right)$$ is the pre-image of U under p.

The sheets of the covering are

$$V_n = \left(n - \tfrac{1}{4}, n + \tfrac{1}{4}\right)$$

The fiber of b is

$$p^{-1}(b) = \{ t \in \mathbb{R} \mid (\cos(2\pi t), \sin(2\pi t)) = b \}$$

where $p : \mathbb{R} \longrightarrow S^1$ defined by

$$p(t) = (\cos(2\pi t), \sin(2\pi t))$$

Also, $p^{-1}(U) = \{ t \in \mathbb{R} \mid \cos 2\pi t > 0 \}$

$$= \bigcup_{n \in \mathbb{Z}} (n - 1/4, n + 1/4)$$

$$n - 1/4 < x < n + 1/4$$

$$2\pi n - \frac{2\pi}{4} < 2\pi x < 2n\pi + 2\pi/4$$

$$-\pi/2 < 2\pi x - 2\pi n < \pi/2$$

$$\cos(2\pi x - 2\pi n) = \cos(2\pi x) > 0$$

$$x \in (n - 1/4, n + 1/4)$$

$$2\pi x \in (2n\pi - \pi/2, 2n\pi + \pi/2)$$

$\overline{V_n} = [n - 1/4, n + 1/4]$

$P|_{\overline{V_n}}$ is injective because $p(t) = (\cos 2\pi t, \sin 2\pi t)$

is injective ~~as~~

Here $\sin 2\pi x$ is strictly monotonic

on $x \in [n - 1/4, n + 1/4]$

$f(x) = \sin 2\pi x$, $f'(x) = \cos 2\pi x . 2\pi > 0$

$P|_{\overline{V_n}} : \overline{V_n} \longrightarrow \overline{U}$ and $P|_{V_n} : V_n \to U$

is an onto map by intermediate theorem.

Therefore P is one-one & onto.

Now use the theorem $f : (X, T_1) \longrightarrow f(X, T)$

$f : (X, T_1) \longrightarrow (Y, T_2)$ be a

continuous bijection. If (X, T_1) is

compact and (Y, T_2) is Hausdorff.

then f is a homeomorphism.