# Topology.

(1) let $\tau \subseteq P(X)$. $\tau$ is topology if its satisfy the 3-properties

(1) $\phi \in \tau$ and $X \in \tau$

(2) For all $U, V \in \tau$, we have $U \cap V \in \tau$

(3) for all $F \subseteq \tau$, we have $\bigcup F \in \tau$

Q.

For every set $U \subseteq X$, we have

if $U$ is a saturated then

$U$ is intersection of open sets and

$X - U$ is a union of closed sets.

Now for any points $x \in X$, we have

$X - \{x\}$ is saturated mean $\{x\}$ is

a union of closed sets, $\Rightarrow$ implies

$\{x\}$ itself is a closed sets.

i.e $$U = \bigcup_{x \in U} \{x\}.$$

$$X - U = X - \bigcup_{x \in U} \{x\}$$

$$= \bigcap_{x \in U} (X - \{x\})$$

Let $A \subset X$ and let $A^c = X - A$

$$A = X - A^c = X \setminus A.$$

$$= X \setminus \bigcup_{x \in A^c} \{x\}$$

$$= \bigcap_{x \in A^c} (X \setminus \{x\})$$

=) 1) intersection of open sets

2) $X \setminus \{x\}$ is open, since $\{x\}$ is closed.

Q. Consider the topological space $(X, T)$, where the set $X = \{a, b, c, d, e\}$, the topology

$$T = \{X, \phi, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}.$$

$A = \{a, b, c\}$.. find the limit points of A.

Soln:

The point a is a limit point of A if and only if every open set containing a contain another points of the set A.

$b, d, e$ are limit points of A.

# Limit points theorem :-

**Limit point Theorem :-**

Let $A$ be a subsets of $X$. Then the point $x$ is a limit point of $A$ if and only if every neighborhood of $x$ contains infinitely many points of $A$.

**proof :-**

**Step 1 :** Suppose $x$ is a limit points of $A$, and suppose some neighborhood $U$ of $x$ intersect $A$ in only finitely many points.

**Step 2 :** Then $U$ also intersects $A - \{x\}$ in finitely many points

**Step 3 :-** Let $\{x_1, x_2 \dots x_m\}$ be the points of $U \cap (A - \{x\})$

Step 4 :- The set $X - \{x_1, x_2 \dots x_m\}$ is an open set $X$, because $\{x_1, x_2 \dots x_m\}$ is closed.

Step 5 :- Now, from step (3), we have

$\{x_1, x_2 \dots x_m\} \in (A - \{x\}) \cap U$

i,e $(A - \{x\}) \cap U = \{x_1, x_2, x_3 \dots x_m\}$

so $\underbrace{(A - \{x\}) \cap U} \cap (X - \{x_1, x_2 \dots x_m\})$

$= \{x_1, x_2 \dots x_m\} \cap (X - \{x_1, x_2 \dots x_m\}$

$= \phi$

i,e $((A - \{x\}) \cap (U)) \cap (X - \{x_1, x_2 \dots x_m))$

$= \phi$

Step (6) $\left(A - \{x\}\right) \cap \left(U \cap \left(X - \{x_1, x_2 .. x_m\}\right)\right)$

$$= \phi$$

i.e $U \cap \left(X - \{x_1, x_2 \dots x_m\}\right)$ doesnot intersect the set $A - \{x\}$

Therefore $U \cap \left(X - \{x_1, x_2 \dots x_m\}\right)$ is a neighborhood of $x$ that intersect the set $A - \{x\}$ not at all.

Now we got contradiction that $x$ is a limit point of $A$.

Therefore $x$ must be limit point of $A$.

Limit point :

let $(X, T)$ be Topological space

let $A \subseteq X$, a point $x \in X$ is a limit point of $A$ if and only if every open neighborhood $U$ of $x$ satisfies, $A \cap (U \setminus \{x\}) \neq \phi$

i/e if and only if every open set $U \in T$ such that $x \in U$ contain some point of $A$ distinct from $x$.

Example : The set of limit point of $\mathbb{Q}$ is $\mathbb{R}$

Since for any $x \in \mathbb{R}$ and any $\varepsilon > 0$, there exist a rational number $r \in \mathbb{Q}$ satisfying $x < r < x + \varepsilon$, $\Rightarrow r \neq x$.

so every open neighborhood $r$ of $x$ satisfy, $\mathbb{Q} \cap (r \setminus \{x\}) \neq \phi$

## Limit point in finite topological space

. Let $(X, T)$ be Topological space.

Let $A \subseteq X$, The point $a \in X$ is a limit point of $A$ if and only if every open set containing $a$ contain another point of the set $A$.

Example :- Take $X = \{a, b, c, d, e\}$

~~T=ϕ~~

$$T = \{X, \phi, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$$

$$A = \{a, b, c\}$$

what is the limit points of $A$ in $(X, T)$?

Ans: Here $T = \begin{cases} X \\ \phi \\ \{a\} \\ \{c, d\} \\ \{a, c, d\} \\ \{b, c, d, e\} \end{cases}$

Here $T = \begin{cases} X \\ \phi \\ \{a\} \\ \{c, d\} \\ \{a, c, d\} \\ \{b, c, d, e\} \end{cases}$

Case I :- $\{a\}$

we know that $\{a\}$ is open in $(X, T)$

But $\{a\}$ contain no other points of

$A$. where $A = \{a, b, c\}$.

So $a$ is not limit point of $A$.

Case II :- $\{c, d\}$

$\{c, d\}$ is open set containing $c$

but not contains the other points of $A$.

So $c$ is not limit points of $A$.

Case III :- $\{a, c, d\}$

Here $\{a, c, d\}$ contain $d$, $\{a, c, d\}$ is an open set.

$d$ contain the other points of $A$.

i,e $a, c$

Therefore $d$ is a limit points of $A$.

Case IV :- $\{b, c, d, e\}$

Here $\{b, c, d, e\}$ contain $e$

$e$ contain the other points of $A$

i,e $b, c$

So $e$ is a limit points of $A$.

Case V : The only open sets containing b are X and {b, c, d, e} and both contain another element of A i.e c so b is a limit point of A. because every open set containing b contains a points of A other than b.

Finally, $\boxed{b, d \text{ and } e}$ are limit point of A in (X, T) Ans.

**Theorem:-** Let $A$ be a subset of a topological space $(X, T)$ and $A'$ the set of all limit points of $A$. Then $A \cup A'$ is a closed set $A$.

**Proof:-** First understand the meaning of $A'$ the set of all limit points of $A$.

**Meaning:-** If $p$ is a limit point of $A'$, then every neighborhood of $p$ contain some point $q \in A'$, and every neighborhood of $q$ contain some point $r \in A$.

Therefore every neighborhood of $p$ contain some point $r \in A$

This implies $p$ is a limit point of $A$,

$\Rightarrow$ i.e $A'$ the set of all limits points of $A$.

Now To prove :- the main theorem.

Step 1 : If $p$ is in $A'$, then every neighborhood of $p$ intersect $A$ then $p \in \bar{A}$,

$A' \subset \bar{A}$ , $A \subset \bar{A}$

so it implies $A \cup A' \subset \bar{A}$

$\Rightarrow$ $A \subseteq A'$

$A \cup A' \subseteq A' \cup A' \subseteq \bar{A}$

so $A \cup A'$ is a closed set.

## Closure :-

let A be a subset of a topological space $(X, T)$. Then the set $A \cup A'$ consisting of A and all its limit points is called the closure of A. it is denoted by $\bar{A}$

$$\boxed{\bar{A} = A \cup A'}$$ Formula.

Q. let $X = \{a, b, c, d, e\}$
and $T = \{X, \phi, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$.

find $\overline{\{b\}}$, $\overline{\{a, c\}}$, and $\overline{\{b, d\}}$ ?

Soln:

→ we know $\bar{A} = A \cup A'$

where $A'$ is the set of limit point of A

i.e $\overline{\{b\}} = \{b\} \cup \{b\}'$

To find the limit point of $\{b\}$,

we have to verify the neighborhood point.

~~Xe~~

~~in $T = \{b, c, d, e\}$~~

in $T = \{X, \phi, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$

Here only $\{b, c, d, e\}$ contain point b

So we will find the limit point of $\{b\}$ from the set $\{b, c, d, e\}$

i.e neighborhood point are $c, d, e$

But Here $c, d$ are not eligible for limit point of $\{b\}$, because ~~c+d~~ in other set like $\{c, d\}$, $\{a, c, d\}$ doesn't contain $\{b\}$

Therefore the limit point of $\{b\}$

$$= \{b\} \cup \{e\}$$

$$\overline{\{b\}} = \{b, e\}$$

To find the limit point of $\{a, c\}$

$$\overline{\{a, c\}} = \{a, c\} \cup \{a, c\}$$

$T = \{X, \phi, \{a\}, \{c,d\}, \{a,c,d\},$

$\{b,c,d,e\}\}.$

$\Rightarrow$ The set $\{a\}, \{c,d\}, \{a,c,d\}$

$\{b,c,d,e\}$ contain $a, c$

$\Rightarrow$ i.e $\{a, c\}$ neighborhood

are $\{b,d,e\}$, so $\{a,c\}' = \{b,d,e\}$

so $\overline{\{a,c\}} = \{a,c\} \cup \{a,c\}'$

$= \{a,c\} \cup \{b,d,e\}$

$= \{a,b,c,d,e\}$

$= X$

$\overline{\{a,c\}} = X$

To find $\overline{\{b, d\}}$,

in $T = \{X, \Phi, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$.

The only set contain $\{b, d\}$ is

$\{b, c, d, e\}$

i.e neighborhood of $\{b, d\}$

i.e ~~d, e~~ c, e

so $\{b, d\}'$ = limit point of $\{b, d\}$

= ~~d, e~~ c, e

$$\overline{\{b, d\}} = \{b, d\} \cup \cancel{\{d, e\}} \{c, e\}$$

$$= \{b, c, d, e\}$$

Q. Given $(X, T)$ is cofinite topology

where $X = \mathbb{Z}$ = set of integer.

i.e ~~(X, Z~~ $(\mathbb{Z}, T)$

① what is the limits of points of set of odd integer $\mathbb{Z}$ in $(X, T)$?

Soln:

we know that in $\mathbb{Z}$,

closed set are two type in $\mathbb{Z}$

① finite

② whole set equivalent to $\mathbb{Z}$

limit points of set ~~of~~ of odd integers

let $A = \{1, 3, 5, 7, \dots\}$

we know ~~Z~~ A is not finite

so A must belong to (2nd category)

i.e A must be whole set $\Rightarrow$

if ~~A~~ ~~wants to be~~

⇒ if A want to become close set.

i/e $\overline{A} = \mathbb{Z}$

Since the only ~~set~~ closed set containing A is $\mathbb{Z}$ i/e limit point is $\mathbb{Z}$.

Q2. What is the interior point of the set of odd integer in $(X, T)$?

Ans: Take $A = \{1, 3, 5, \ldots\}$

$A^c = \{$ even integer$\}$

$A^c =$ infinite

so A is not open in cofinite topology.

A must b finite, if A want to become open set in $(X, T)$.

so $\boxed{A^\circ = \phi}$

## Subspace Topology / Relative Topology / INDUCED TOPOLOGY :

⇒ let $(X, T)$ be a topological space. let $Y \subseteq X$ be any ~~subset~~ non-empty subsets of a topological space $(X, T)$.

The collection $T_Y = \{U \cap Y : U \in T\}$ of ~~sett~~ subsets of $Y$ is a topology on $Y$ called the subspace Topology.

i.e $(Y, T_Y)$ is a subspace of $(X, T)$

Example :- let $X = \{a, b, c, d, e, f\}$

$T = \{X, \phi, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e, f\}\}$

and $Y = \{b, c, e\}$. Then the subspace Topology on $Y$ will be

→ $Y \cap X = Y$

→ $Y \cap \{a\} = \phi$

$Y \cap \{c, d\} = \{c\}$.

$Y \cap \{a, c, d\} = \{c\}$.

$Y \cap \{b, c, d, e, f\} = \{c\}$.

so $T_Y = \{Y, \phi, \{c\}\}$.

Q. # Consider the subset $[1, 2] \subseteq \mathbb{R}$, A basis for the subspace Topology $T$ on $[1, 2]$ is

$A = \{[a, b) \cap [1, 2] : a, b \in \mathbb{R}, a < b\}$

$A = \{[a, b) \cap [1, 2] : a, b \in \mathbb{R}, a < b\}$

$= \cancel{\{(a,b) : 1 \leq a < b \leq 2\} \cup \{[1, b) : 1 \leq b \leq 2}$

$= \{[a, b) : 1 \leq a < b \leq 2\} \cup \{[1, b) : 1 < b \leq 2\}$

$\cup \{(a, 2] : 1 \leq a < 2\} \cup \{[1, 2]\}$

is a basis for $T$.

Q. Prove that $(\mathbb{Z}, T_{subspace})$

$= (\mathbb{Z}, T_{discrete})$ ?

proof: Here $(\mathbb{Z}, T_{subspace})$

= Subspace Topology

$(\mathbb{Z}, T_{discrete})$ = Discrete Topology

proof: Let $n \in \mathbb{Z}$. Then $\{n\} = (n-1, n+1) \cap \mathbb{Z}$.

we know that $(n-1, n+1)$ is open in $\mathbb{R}$

and therefore $\{n\}$ is open in the Subspace Topology on $\mathbb{Z}$.

Thus every singleton set in $\mathbb{Z}$ is open in the subspace topology on $\mathbb{Z}$.

so the Subspace Topology $(\mathbb{Z}, T_{subspace})$

= Discrete topology $(\mathbb{Z}, T_D)$

Since in discrete topology every point is isolated.

## Continuous Function :-

$f: \mathbb{R} \to \mathbb{R}$ is continuous if and only if for each $a \in \mathbb{R}$ and each interval $(f(a) - \varepsilon, f(a) + \varepsilon)$ for $\varepsilon > 0$, there exist a $\delta > 0$ such that

$f(x) \in (f(a) - \varepsilon, f(a) + \varepsilon)$ for all $x \in (a - \delta, a + \delta)$

Q. Show that the function

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

is continuous at $r = 0$ and discontinuous at every $r \neq 0$.

proof:- choose $\varepsilon > 0$, Choose $\delta = \varepsilon$. Then

if $|x| < \delta = \varepsilon$, we have

$$|f(x) - f(0)| = |f(x)| = |x| < \varepsilon$$

if $x$ is rational.

$|f(x) - f(0)| = 0 < \varepsilon$ if $x$ is irrational.

$|x| < \delta$ implies $|f(x) - f(0)| < \varepsilon$ if $x$ is irrational.

In either case, $|x| < \delta$ implies $|f(x) - f(0)| < \varepsilon$.

This prove continuity at $r = 0$.

Now let $r \neq 0$. Take a sequence $\{x_n\}$ of rationals converging to $r$.

Then $f(x_n) = x_n \longrightarrow r$.

Also take a sequence $\{y_n\}$ of irrational converging to $r$.

then $f(y_n) = 0$, since $r \neq 0$, it implies that ~~lim~~ $\lim_{r+} f \neq \lim_{r-} f$

i.e at $r$ limit of $f$ doesnot exist.

Hence $f$ is not continuous at $r$.

Metrizable space :- A topological space $(X, T)$ is said to be metrizable if there is a metric d on X that generates T.

# Indiscrete topology is not ~~Haus~~ metrizable?

Proof :- Take $X = \{0, 1\}$, $T = \{\phi, X\}$.

$(X, T)$.

~~Here~~ we know that every metric space is Hausdorff.

But Here indiscrete topology is not Hausdorff.

Hausdorff space definition :- $(X, T)$

① if for any point ~~and~~ $x, y$ in X

② we can find two set U and V in T such that $x \in U$, $y \in V$, $U \cap V = \phi$

Take $X = \{0, 1\}$, $T = \{\phi, X\}$.

$x = 0$, $\cancel{\&}$ $\cancel{y = 0}$ $y = 1$.

$\phi$ contain no point

Take two point $x$ and $y$

where $x = 0$, $y = 1$.

So there $x \in U$
$y \in U$

$$U \cap U \neq \phi$$

i,e we can not separate the point

i,e there can not be any $\varepsilon$ - balls

separating $x$ from $y$

proof:- Take $x, y \in X$ and $r = d(x, y)$, then

we have $B(r, x) = \{x\}$.

i,e $B(r, x) \neq X$

we know that in indiscrete topology the only open set are $X$ and $\phi$

Take $r = 1/2$, $x = 1$.

$B(1/2, 1) = \{1\} \neq X$

so $B(1/2, 1)$ is not open.

# To separate them, you would need an open ~~sets~~ $U$ containing $x$ and an open set $V$ containing ~~to~~ $y$, such that $U \cap V = \phi$.
However, the only open set available is $X$ itself.
If you choose $U = X$ and $V = X$, their intersection is still $X$ which is not the empty set.

Normal space:- A topological space $(X, T)$ is said to be a normal if for each pair of disjoint closed set A and B, there exist open set U and V such that $A \subseteq U$, $B \subseteq U$ and $U \cap V = \phi$.

Q. Prove that every metrizable space is a normal space?

soln:

Step 1: Suppose A, B disjoint closed subset in a metric space $(X, d)$.

Step 2:- Take a point $a \in A$, and open ball $B(a, r_a) \subset A$

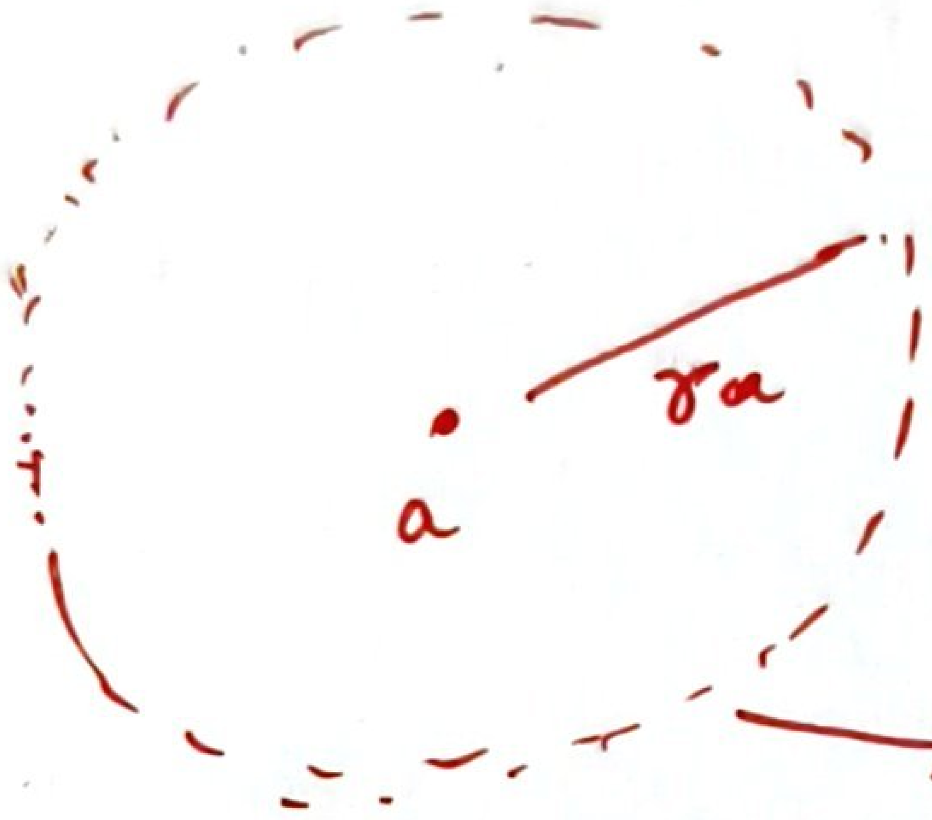
i.e $B_d(a, r_a) = \{x \in X : d(x, a) < r_a\}$

$= \{x \in X : |x - a| < r_a\}$

such that

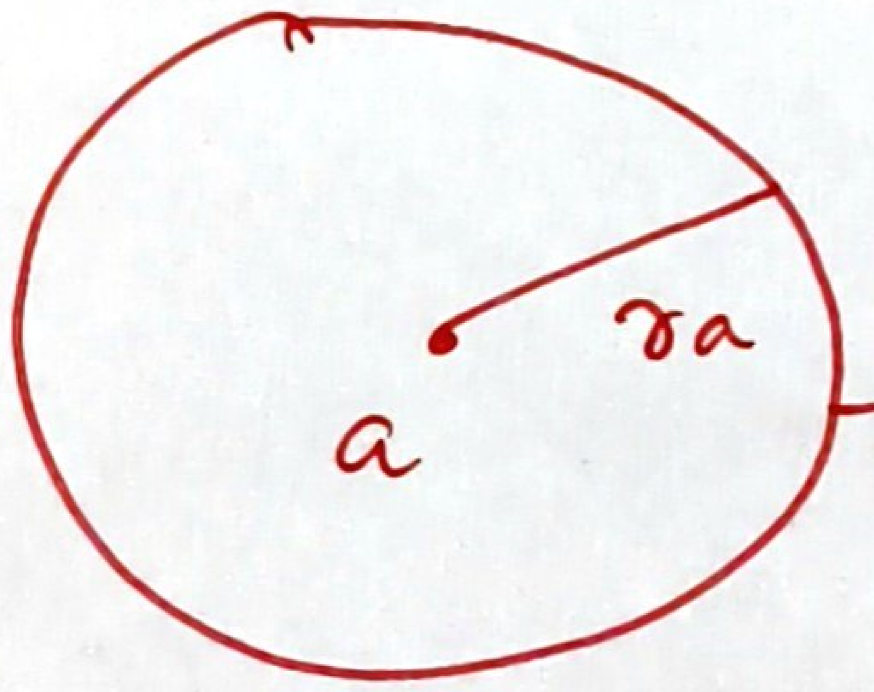
$B(a, r_a) \cap B = \phi$.

closed ball.




$\rightarrow B(a, r_a). = $ open ball

$\rightarrow$ ~~opp~~ open ball




$\overline{B(a, r_a)}$ = closed ball

$\Rightarrow B(a, r_a) \cap B = \phi$ mean $a \notin B$.

where $B$ is closed since complement of open set is closed.

~~Step 3~~ **Step (3)** Now we have for all $a \in A$, there exist $b \in B$ such that $d(a, b) \geqslant r_a$.

Step 4 :- Similarly, take ~~b ∈ B~~, point $b \in B$

$$B(b, r_b) \subset B,$$

$$B(b, r_b) \cap A = \phi$$

i.e for all $b \in B$ there exist $a \in A$ such that $d(b, a) \geq r_b$.

Step 5 :- Now define $U = \cup \{ B(a, \frac{r_a}{2}) : a \in A \}$

$$V = \cup \{ B(b, \frac{r_b}{2}) : b \in B \}$$

which are both open (since union of open set is open)

So, we have $A \subseteq U$, $B \subseteq V$.

Step 6 :- suppose $z \in U \cap V$ which mean that for some point ~~we have~~ $a_0 \in A$ and $b_0 \in B$, we have

$z \in B(a_0, \frac{r_{a_0}}{2})$ and ~~$z \in B(b_0, r_b$~~

$$z \in B\left(b_0, \frac{r_{b_0}}{2}\right).$$

Step (6) Now take $\boxed{r = \max\{(r_{a_0}, r_{b_0})\}}$

From step (3), we have

$$\boxed{d(a,b) \geq r_a.}$$

So we have $d(a_0, b_0) \geq r$.

Step (7) Now $r \leq d(a_0, b_0) \leq d(a_0, z) + d(z, b_0)$

Triangle inequality

$$\Rightarrow r \leq d(a_0, b_0) \leq d(a_0, z) + d(z, b_0)$$

$$< \frac{r_0}{2} + \frac{r_b}{2}.$$

$$\leq \frac{r}{2} + \frac{r}{2}$$

$$= r$$

$r < r$ ( Contradiction ).

it is impossible

actuall $r = r$ (possible)

so we are getting contradiction

Therefore $U \cap V = \phi$,

This prove that every metrizable space $X$

is a normal space.

## Bounded sets :-

let $(X,d)$ be a metric space and let $Y$ be a subset of $X$. i.e $Y \subseteq X$

$\Rightarrow$ Then $Y$ is bounded if there exist a ball $B(x,r)$ in $X$ such that $Y \subseteq B(x,r)$.

## Totally Bounded :-

~~let~~ $(X,d)$ is said to be totally bounded if, for every $\varepsilon > 0$, one can cover $X$ by a finite number of open balls of radius $\varepsilon$.

i.e if for each $\varepsilon > 0$, there exist $x_1, x_2, x_3, \ldots, x_n$ in $X$ such that the number of finite ball $B(x_1, \varepsilon), B(x_2, \varepsilon), B(x_3, \varepsilon) \ldots B(x_n, \varepsilon)$ ~~which~~ which cover $X$.

$$\text{Here } \boxed{X = \bigcup_{i=1}^{n} B(x_i, \varepsilon)}$$

Q. Prove that every totally bounded metric space is a bounded metric space.

Soln: let $(X, d)$ be a totally bounded metric space.

Step 1: For every $\varepsilon > 0$, there exist a finite number of ~~bat~~ balls $B(x_1, \varepsilon)$, $B(x_2, \varepsilon)$, ... $B(x_n, \varepsilon)$ such that

$$X = \bigcup_{i=1}^{n} B(x_i, \varepsilon)$$

Step 2: we have to show that $X$ is bounded, i.e. there will exist $M > 0$ such that

$$X \subseteq B(x_0, M)$$

Step 3 :- Take two points $x$ and $y$,

$x, y \in X$. Here $X = \bigcup_{i=1}^{n} B(x_i, \varepsilon)$,

then there exist $B(x_m, \varepsilon)$ and

$B(x_k, \varepsilon)$ such that

$x \in B(x_m, \varepsilon)$ and $y \in B(x_k, \varepsilon)$

for ~~$1 \leq k \leq m \leq n$~~. $1 \leq k, m \leq n$.

By triangle inequality.

$$d(x,y) \leq d(x, x_m) + d(x_m, x_k) + d(x_k, y)$$

Since ~~$d(a,b) \leq \varepsilon$~~ $d(a,b) \leq \varepsilon$

$$\Rightarrow d(x,y) \leq \varepsilon + d(x_m, x_k) + \varepsilon.$$

$$\boxed{d(x,y) \leq 2\varepsilon + d(x_m, x_k)}$$

Since X is covered by a finite number of ball. We can take the maximum of the distance between the centres of the ball and boundary of ball

Now, we have

$$\boxed{d(x_m, x_k) \leq \max_{1 \leq i, j \leq n} \{d(x_i, x_j)\}}$$

Step (4) :- Then from step (3), we have

$$d(x,y) \leq 2\varepsilon + \max_{1 \leq i, j \leq n} \{d(x_i, x_j)\}.$$

Take $M = 2\varepsilon + \max_{1 \leq i, j \leq n} \{d(x_i, x_j)\}$

$\boxed{d(x,y) \leq M}$ where $M > 0$ and $\forall\, x, y \in X$.

Therefore $(X, d)$ is a bounded Metric space.

Hence proved

**Open ball** :- Suppose $(X, d)$ be a metric space and $x \in X$. For each $r \in \mathbb{R}^+$ (positive real number). we ~~have~~ define

$$B(x, r) = \{ y \in X : d(x, y) < r \}$$

$B(x, r)$ meaning open ball in $X$ centre at the point $x$ and radius $r$.

** in Real line, in ($\mathbb{R}$)

$$B(x, r) = (x - r, x + r)$$

i/e open balls of $\mathbb{R}$ with the usual metrics are the open interval of ~~to~~ the type $(a, b)$ where $a, b \in \mathbb{R}$ with $a < b$

i/e $(a, b) = B\left(\frac{a+b}{2}, \frac{b-a}{2}\right) = B\left(\frac{a+b}{2}, \frac{b-a}{2}\right)$

$$= \left(\frac{a+b}{2} - \frac{b+a}{2}\right), \left(\frac{a+b}{2} + \frac{b-a}{2}\right)$$

Q. prove that $\mathbb{R}$ with Euclidean metric is not ~~totae~~ totally bounded?

Soln: Consider an open ball

$$B(x, r)$$

By the definition of totally bounded

if for each $\varepsilon > 0$, there exist $x_1, x_2, \ldots x_n$ in $X$ such that $\mathbb{R} \subseteq \bigcup_{i=1}^{n} B(x_i, \varepsilon)$

We know in Real line ($\mathbb{R}$), the open ball are open interval.

So let us take an open interval

$$B(x_i, r) = (x_i - r, x_i + r)$$

for some $x_i \in \mathbb{R}$ and $r > 0$.

Now if $(\mathbb{R}, d)$ is totally bounded, then there would exist $x_1, x_2, \dots x_N \in \mathbb{R}$ such that

$$R \subseteq \bigcup_{i=1}^{N} (x_i - r, x_i + r) = \bigcup_{i=1}^{N} B(x_i, r).$$

$$\mathbb{R} \subseteq \bigcup_{i=1}^{N} B(x_i, r)$$

Since $\mathbb{R}$ is infinite

and $\bigcup_{i=1}^{N} B(x, r)$

not possible that $\boxed{(\text{infinite}) \subseteq \text{finite.}}$

we are getting contradiction,

Therefore $\mathbb{R}$ with Euclidean metric

is not totally bounded.

## Quotient topology :

let $X$ be a topological space and let

$\sim$ be an equivalence relation on $X$.

The quotient set $Y = X/\sim$ is the set

of equivalence classes of elements of $X$.

let $q : X \longrightarrow Y$ be an onto map defined

by $q(x) = [x]$

Then collection $\tau = \{ V \subseteq Y \mid q^{-1}(V)$ is open in $X \}$

$\Rightarrow \tau$ turns out to be a topology on $Y$

$\tau$ is a family of subset of ~~X~~ $Y$

This topology $\tau$ is said to be the quotient topology on $Y$ and such a map $q$ is said to be the quotient map.

or

$\tau$ is said to be quotient topology on $Y$ ~~and~~ induced by the quotient map $q$.

<u>Note</u>: Sometime quotient map is called identification map

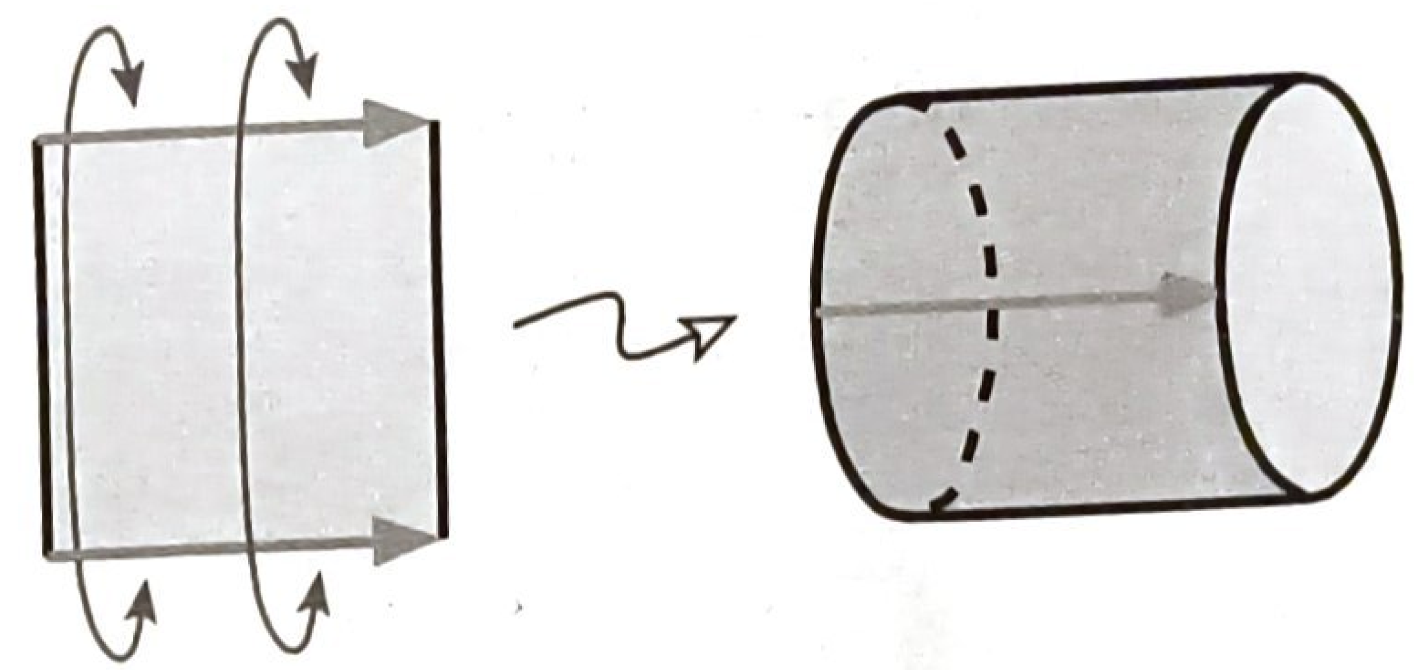
# Quotient Topology

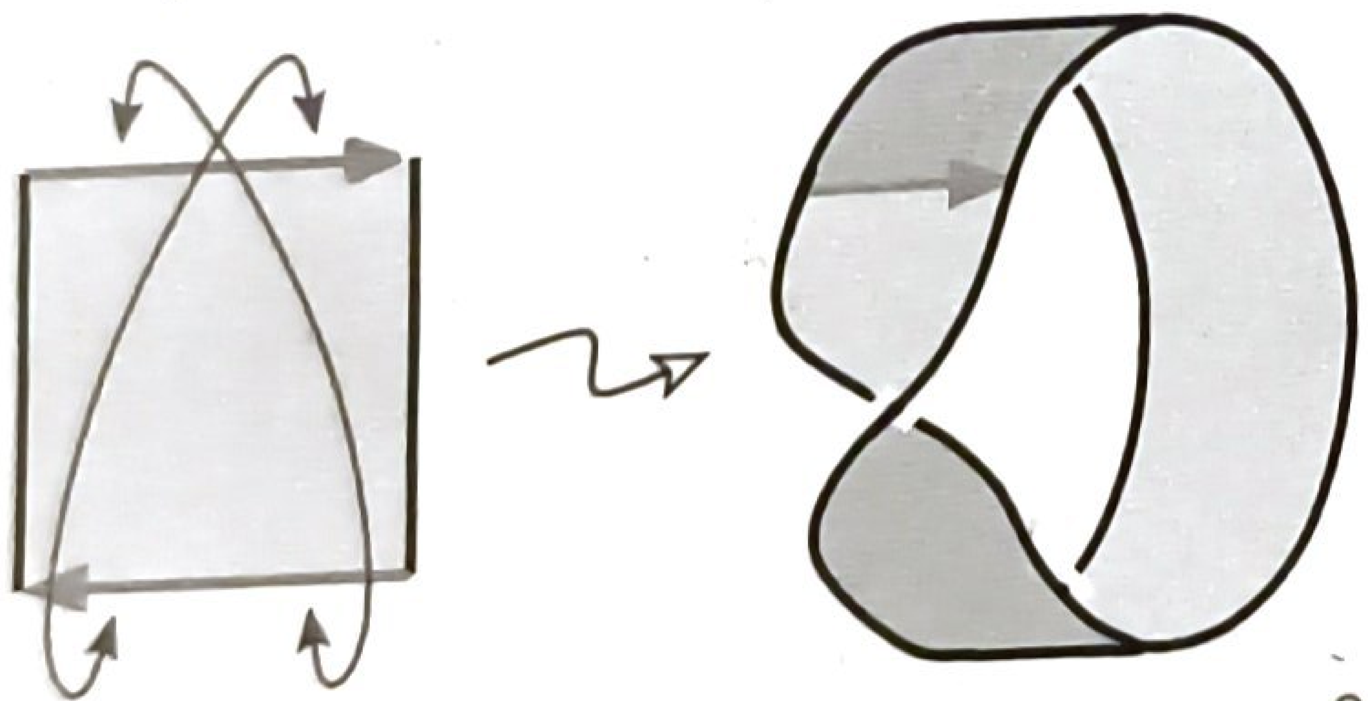
**Definition.** Let $X$ be a topological space with an equivalence relation $\sim$ which is *reflexive* ($x \sim x$), *symmetric* ($x \sim y$ implies $y \sim x$), and *transitive* ($x \sim y$ and $y \sim z$ imply $x \sim z$). The set of equivalence classes $Y := X/\sim$ can be endowed with *quotient topology* whose open sets are the subsets of $Y$ which preimages under the canonical projection $X \to X/\sim$ are open sets in $X$.

**Examples.**

(1) Let $X = I := [0,1]$ be the unit interval with the equivalence relation, $0 \sim 1$, gluing the end-points. The quotient space $X/\sim$ is a circle $S^1$.

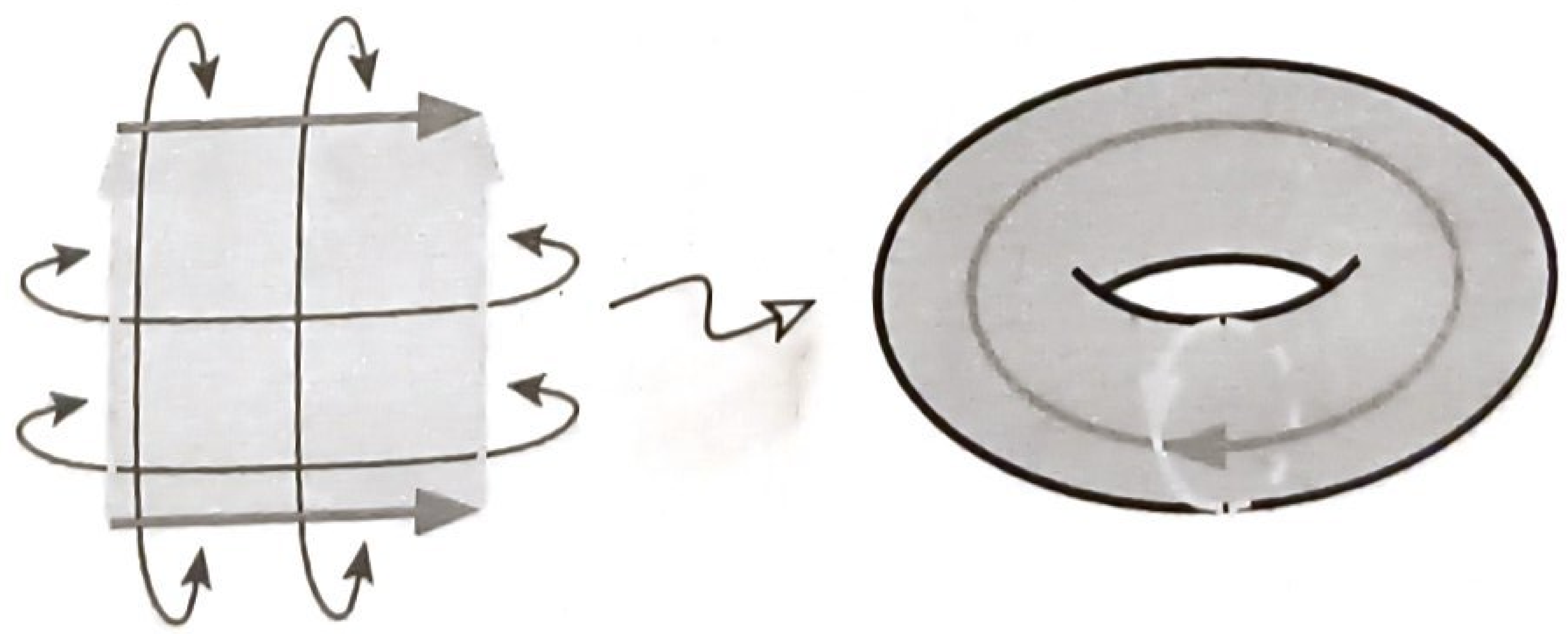

(2) Let $X = I^2 := [0,1] \times [0,1]$ be the unit square with the equivalence relation, $(t,0) \sim (t,1)$ for all $0 \leqslant t \leqslant 1$, gluing the opposite sides. The quotient space $X/\sim$ is a cylinder.



(3) 

Let $X = I^2$ be the unit square with the equivalence relation, $(t,0) \sim (1-t,1)$ for all $0 \leqslant t \leqslant 1$, gluing the opposite sides reversing the orientation. The quotient space $X/\sim$ is the *Möbius band*.

(4) Let $X = I^2$ be the unit square with the equivalence relation, $(t,0) \sim (t,1)$ and $(0,t) \sim (1,t)$ for all $0 \leqslant t \leqslant 1$, gluing the opposite sides in pairs. The quotient space $X/\sim$ is the *torus*.



# Discrete topology

The discrete topology T is the largest possible topology on a set X, as it includes every every possible subset of X.

. The discrete topology on a set X is the topology given by the power set of X.
. That is, every subset of X is open in the discrete topology.

A space equipped with the discrete topology is called a discrete space.

let's consider a simple example of discrete topology using a finite set X consisting of three elements.

$X = \{a, b, c\}$

The power set of X, which is the set of all possible subsets of X, comprises the following subsets:

1. The empty set : $\phi$
2. Subsets with individual elements : $\{a\}, \{b\}, \{c\}$
3. Subsets with two elements : $\{a, b\}, \{a, c\}, \{b, c\}$
4. The entire set $\{a, b, c\}$

In the discrete topology on X, every possible subset of X is considered an open set.
Therefore, the discrete topology T on X is given by

$$T = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$$

Take the subset $\{a\}$. By the definition of discrete topology, it is an open set. Simultaneously, this subset $\{a\}$ is closed because its complement. $X \setminus \{a\} = \{b, c\}$ is an open set.

So in discrete topology, the subset $\{a\}$ is both open and closed.

## Metrizable :

A topological space $(X, T)$ is said to be metrizable if there is a metric $d$ on $X$ that generates $T$.

Every discrete topological space is metrizable because the discrete topology on $X$ is ~~t~~ induced by the metric

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

proof :- let $x \in X$ where $X$ is a set and let $0 < r \leq 1$, then $B_d(x, r) = \{x\}$, so $\{x\}$ is an open set.

If $A \subseteq X$, then $A = \bigcup_{x \in A} \{x\}$ is a union of open sets and therefore $A$ is open since union of open sets is open.

Every subset of $X$ is open and $X$ has the discrete topology.

# Show that in a discrete metric space, every subset is both open and closed.

Soln.

let X be a non-empty set. Define a map on $X \times X$ by

$$d(x,y) = \begin{cases} 0 & , x = y \\ 1 & , x \neq y \end{cases}$$

Then d is a metric on X called the discrete metric

let $x \in X$. Consider an open ball around x of radius r in X.

$$B(x,r) = \{ y \in X : d(x,y) < r \}$$

if $r \leq 1$, then

$$B(x,r)|_{r=1} = \{ y \in \mathbb{R} : d(x,y) < 1 \}$$

If $d(x,y) < 1 \Rightarrow d(x,y) = 0$

$x = y$

$$\Rightarrow B(x,1) = \{ y \in \mathbb{R} : d(x,y) = 0 \}$$

$$B(x,1) = \{ x = y \in \mathbb{R} : d(x,x) = 0 \}$$