Q. If $A \in M_{n\times n}(\mathbb{F})$ is invertible, then for some $b \in \mathbb{R}^n$, the system $Ax = b$ is inconsistent.

True | False

Sol: False, we always have $x = A^{-1}b$ as a solution if A is invertible

Q. Find ~~the~~ bases for the following subspaces of $M_{3\times3}(\mathbb{F})$.

$A = \{B \in M_{3\times3}(\mathbb{F}) \mid B^T = -B\}$.

Sol.

A basis for A is

$$S = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \right\}$$

Q. If $A = \begin{pmatrix} \frac{3}{4} & 1/4 \\ 1/4 & 3/4 \end{pmatrix}$. find an expression for A^n, where n is an arbitrary positive integer. What is $\lim_{n \to \infty} A^n$?

Sol. $|A - \lambda I| = 0$

$\Rightarrow 2\lambda^2 - 3\lambda + 1 = 0$

$\Rightarrow (2\lambda - 1)(\lambda - 1) = 0$

$\cancel{\lambda_1 = 1/2}$, $\lambda = 1/2, 1$

let $\lambda_1 = 1/2$ and $\lambda_2 = 1$

$A - \lambda_1 I = \begin{pmatrix} 1/4 & 1/4 \\ 1/4 & 1/4 \end{pmatrix}$ so $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is a corresponding eigenvector.

likewise $A - \lambda_2 I = \begin{pmatrix} -1/4 & 1/4 \\ 1/4 & -1/4 \end{pmatrix}$ so

$v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector for λ_2.

Define $P = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix}$

we then have,

$A = PDP^{-1}$ so, for any integer n,

$$A^n = PD^nP^{-1}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1/2^n & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\lim_{n\to\infty} A^n = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Q. If S is a linearly dependent set of vectors, then some non-empty subset of S has to be linearly independent. True / False

Ans. If $S = \{0\}$, then S is linearly dependent and the only non-empty subset of S is itself.

So (False)

Q. If S_1 and S_2 are disjoint subsets of a vector space V, then $span(S_1) \cap span(S_2) = \{0\}$. True / False

Ans. False.

Let $V = \mathbb{R}^2$, $S_1 = \{(1,0), (0,1)\}$

$S_2 = \{(1,1)\}$

then $S_1 \cap S_2 = \phi$ but

$span(S_1) \cap span(S_2) = \{(t,t) \mid t \in \mathbb{R}\}$

linearly dependent :-

A set of vectors is linearly dependent if there is a non-trivial linear combination of the vectors that equal 0.

linearly independent :-

A set of vectors is linearly independent if the only linear combination of the vectors that equal 0 is the trivial linear combination (i.e all coefficients = 0).

<u>Q</u>: let S be a subset of a vector space V. If v ∈ span(S), then S ∪ {v} is linearly dependent. <u>True / False</u>

<u>Ans</u>. True

v ∈ span(S) ⇒ v can be written as a linear combination of elements of S

so S ∪ {v} is linearly dependent.

Q. If S is a subset of a vector space V and $span(S) \subset S$, then S is a subspace of V.

True / False :

True

Sol. We always have $S \subset span(S)$; having $span(S) \subset S$ in addition implies that $span(S) = S$ so S is a subspace of V.

Q. A subset $S = \{u_1, u_2, u_3\}$ of a vector space V is linearly dependent if and only if any two of the elements of S are multiples of each other. True / False

Sol. False

let $V = \mathbb{R}^2$ and $S = \{(1,0), (0,1), (1,1)\}$

Then, S is linearly dependent but no two of the elements of S are multiples of each other.

Q. Any set that contains the zero vector is linearly dependent. True/False

Sol. True, take $c_1 . 0 = 0$ where $c_1 \in \mathbb{R}$

As $1 . 0 = 0$, we have a non-trivial linear combination of the elements of this set equal to 0.

Q. If $T : V \longrightarrow W$ is linear, then T carries linearly dependent sets in V to linearly dependent set in W. True/False

Sol. True, T preserves linear combination.

Q. If $T : V \longrightarrow W$ is linear, then T carries linearly independent set in V to linearly independent sets in W. True/False

Sol. False, T could be the zero map.

Q. For any linear transformation

~~T : U~~ $T : V \longrightarrow W$ and $U : V \longrightarrow W$

we have $\ker(T) \cap \ker(U) \subset \ker(T+W)$

True / False

Sol. True. If $T(x) = 0_W$ and $U(x) = 0_W$

then $(T+U)(x) = 0_W$.

Q. If $T : V \longrightarrow V$ is a linear transformation,

then we always have $\ker T \cap \operatorname{Im}(T) = \{0\}$.

True / False

Sol. False. Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by

$T(x_1, x_2) = (0, x_1)$. Then

$\ker T = \operatorname{Im}(T) = \{ (0, c) \mid c \in \mathbb{R} \}$

Q. Let $A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a linear transformation such that $\langle Ax, x\rangle = 0$ for all $x \in \mathbb{R}^n$. Then trace $A = 0$.

True/False

Sol True

$$\langle Ax, x\rangle = x^T A x = 0$$

$$\Rightarrow \quad x^T A x = 0$$

Multiply by 2 both sides, we get

$$2x^T Ax = 2.0$$

$$\Rightarrow 2x^T Ax = 0$$

$$\Rightarrow x^T Ax + x^T Ax = 0$$

$$\Rightarrow x^T Ax = -x^T Ax$$

$$\Rightarrow x^T Ax = x^T(-A)x \quad \text{—(1)}$$

$x^T A x$ is a scalar, so

$$x^T Ax = (x^T Ax)^T \quad \text{—(2)}$$

$$= x^T A^T x$$

From (1) and (2), we have

$$x^T A^T x = -x^T A x$$

$$\Rightarrow A^T = -A \quad \xrightarrow{\text{where}} A \text{ is skew-symmetric}$$

or

A is skew-Hermitian if $A \in \mathbb{C}^n$

$A^H = -A$ where A^H denotes the conjugate transpose of the matrix A.

$$A^T = -A \Rightarrow A = -A^T$$

$$\text{trace } A = \text{trace}(A^T)$$

$$\text{trace } A = \text{trace}(-A)$$

$$\Rightarrow 2 \text{ trace } A = 0$$

$$\boxed{\text{trace } A = 0}$$

or

let $A = [a_{ij}]_{n \times n}$. Then $a_{ij} = -a_{ji}$ for all i, j

so $a_{ii} = -a_{ii}$

$2a_{ii} = 0$

$a_{ii} = 0$

$\Rightarrow$ diagonal elements of A are zero.

$\Rightarrow$ Trace $A = 0$

Note :- For eigenvalue of skew symmetric or skew-Hermitian, we have

$$\lambda = -(\bar{\lambda})^T$$

$$\lambda = -\bar{\lambda}$$

$\Rightarrow$ λ is either 0 or pure imaginary number.

Q. If A and B are matrices such that $AB = B$ and B is not the zero matrix, then $A = I$. True / False

Sol. False, take $A = B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

Q. Let U, V, W be finite dimensional spaces and let $T: U \to V$ and $S: V \to W$ be linear. If T is onto then

$\operatorname{Im}(S \circ T) = \operatorname{Im}(S)$. True / False

Sol. True

if T is onto then

$T(U) = V$

$\Rightarrow$ $\cancel{S \circ (T(U))}$ $(S \circ T(U)) = S(V) \Rightarrow$ $\quad S: V \to W$

$\Rightarrow \operatorname{Im} S \circ T = \operatorname{Im}(S(V))$

$= \operatorname{Im}(S)$

Q. The matrices $A = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 5 \\ 1 & 0 \end{pmatrix}$

are similar. True / False

Sol. False

$\text{trace}(A) = 0$

$\text{trace}(B) = 2$

$\text{trace}(A) \neq \text{trace}(B)$

Q. If we define the subspace $V \subseteq \mathbb{R}^3$ by

$V = \{(x, y, z) : x = 2y = -4z\}$, then

V is isomorphic to $\mathbb{R}^2$. True / False

Sol. False, $\dim(V) = 1 \neq 2 = \dim(\mathbb{R}^2)$

Q. The matrices $A, B \in M_{n \times n}(\mathbb{F})$ are called similar if there exist some

$Q \in M_{n \times n}(\mathbb{F})$ such that $B = Q^T A Q$.

True / False

Sol. False, because $B = Q^{-1} A Q$

Q. The set of solutions to a system of linear equations $Ax = b$ is a vector space True / false

Sln False, never true when $b \neq 0$.

Q. A linear operator T on a finite-dimensional space is diagonalizable only its ~~charact~~ characteristic polynomial splits. True / false

Sln True, the splitting of the characteristic polynomial is a necessary condition for ~~diagonal~~ diagonalizability.

Q. If $W_1 \oplus W_2$ is a T-invariant subspace of a vector space V, then W_1 and W_2 are also T-invariant. True / False

Soln False.

$$V = W_1 \oplus W_2 \Rightarrow W_1 \cap W_2 = \{0\}$$

Let $V = \mathbb{R}^2$, $T(x,y) = (-y,x)$

$$W_1 = \{(0,y) \in \mathbb{R}^2 \mid y \in \mathbb{R}\}$$

$$W_2 = \{(x,0) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$$

$$T(W_1) + T(W_2) = T(W_1 + W_2) = \mathbb{R}^2$$

$\Rightarrow W_1 \oplus W_2$ is a T-invariant subspace of a vector space V.

But $T(W_1) = T(0,y) = (-y,0) \notin W_1$

$T(W_2) = T(x,0) = (0,x) \notin W_2$

$\Rightarrow W_1$ and W_2 are not T-invariant

T-cyclic subspace :-

Suppose that $T: V \longrightarrow V$ is a linear transformation, and let $x \in V$. Then

$W = \text{span}(\{x, T(x), T^2(x), \ldots\})$ is a

T-invariant subspace.

Also, W is called the T-cyclic subspace of V generated by x.

<u>proof:-</u> let $y \in W \Rightarrow y = T^n(x)$

Then $T(y) = T(T^n(x)) = T^{n+1}(x) \in W$

$\Rightarrow$ W is a T-invariant subspace.

<u>Example:-</u> let $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be given by

$T(a, b, c) = (2a, a+b, 0)$.

find the T-cyclic subspace of V generated by e_1.

<u>Soln.</u> $T(e_1) = (2, 1, 0)$

$T^2(e_1) = T(T(e_1)) = T(2, 1, 0)$

$= (4, 3, 0) \ldots\ldots$

... so on.

$T^2(e_1)$ is a linear combination of e_1, $T(e_1)$.

Similarly for any n, $T^n(e_1) = (a_1, a_2, 0)$ for some a_1, a_2 and so it is a linear combination of e_1 and $T(e_1)$.

It follows, that the T-cyclic subspace of V generated by e_1 is $\text{span}(\{e_1, T(e_1)\})$

i.e. $\text{span}(\{e_1, T(e_1)\}) = \{(a_1, a_2, 0) \mid a_1, a_2 \in \mathbb{R}\}$

$= \text{span}(\{e_1, e_2\})$

$\Rightarrow$ The T-cyclic subspace of V generated by ~~x~~ e_1 is the subspace of V spanned by the set of all $T^n x$ for n a natural number.

Q. Let $F = \mathbb{R}$ let V be the subspace of polynomials with coefficients in R, and let $T: V \longrightarrow V$ be the operator sending f to xf', where f' is the derivative of f. Let W be the T-cyclic subspace of V generated by x^2+1. Compute the characteristic polynomial of $T|_W$.

Sol $T: V \longrightarrow V$ defined by

$$T(f) = x f' = x \frac{\partial f}{\partial x}.$$

$$W = \text{span}\{f, T(f), T^2(f) \dots\}$$

$$T(f) = T(x^2+1) = x \cdot \frac{\partial}{\partial x}(x^2+1)$$

$$= x \cdot (2x)$$

$$= 2x^2 \quad \text{where } f = x^2+1$$

$$\Rightarrow T^2(f) = T(T(f)) = T(2x^2) = 4x^2$$

$\vdots$

so on.

$\Rightarrow$ $T^2(f)$ is a linear combination

of x^2+1, $T(f) = T(x^2+1)$

i.e $T^2(f)$ is a ~~linear~~ linear combination

of x^2+1 ~~and $2x^2$~~ and $2x^2$

Similarly for $T^n(f)$.

$\Rightarrow$ $W = \text{span}\{x^2+1, 2x^2\}$

$\dim(W) = 2 \Rightarrow 2\times 2$ matrix form

So $T(x^2+1) = 2x^2 = 0.(x^2+1) + 1.2x^2$

$\cancel{T(T(x)}\ T(T(x^2+1)) = T(2x^2)$

$= 0.(x^2+1) + 2.2x^2$

$\Rightarrow T(x^2+1) = 0.(x^2+1) + 1.2x^2$

$\Rightarrow T(2x^2) = 0.(x^2+1) + 2.2x^2$

columns of T are $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$

$$\Rightarrow \quad [T] = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}$$

$$\Rightarrow \quad ch_{T|_W}(x) = -x(2-x)$$
$$= x(x-2)$$
$$= x^2 - 2x$$

Hence the characteristic polynomial of $T|_W$ is $x^2 - 2x$

Q. If x and y are linearly dependent vectors in an inner product space, then $\|x+y\|^2 \geq \|x\|^2 + \|y\|^2$. True / False

Sol. False.

$x = (1,0)$ and $y = (-1,0)$

Q. $\langle f, g \rangle = \int_0^1 f(t) g(t)\, dt$ is an inner product on $C[-1,1]$. True / False

Sol False, doesnot ~~satisfy~~ satisfy the positive condition.

Q. If x and y are vectors in an inner product, then $\|x+y\|^2 \leq \|x\|^2 + \|y\|^2$. True / False

Sol False.

Take $x = y = (1,0)$

Q. $\langle f, g \rangle = \int_0^1 f'(t) g(t)\, dt$ is an inner product on $\mathbb{P}(\mathbb{R})$. True / False

Ans False $f(t) \equiv 1$, $f'(t) = 0$

$\Rightarrow \langle f, f \rangle = 0 = \int_0^1 f'(t) f(t)\, dt$

$\downarrow$

property doesn't hold.

Q. The product of two self-adjoint operators is always self adjoint. True / False

Sol Self-adjoint Matrix (symmetric matrix or Hermitian ~~matr~~ matrix.)

~~True~~ False, take $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

Q. let $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a linear transformation defined by

$$T(x_1, x_2, x_3) = (3x_1 + x_3, -2x_1 + x_2, -x_1 + 2x_2 + 4x_3)$$

find the matrix of T in the standard & ordered basis for $\mathbb{R}^3$.

Sol.

$$T(1,0,0) = (3,-2,-1)$$

$$T(0,1,0) = (0,1,2)$$

$$T(0,0,1) = (1,0,4)$$

$$\Rightarrow \quad [T]_\alpha = \begin{pmatrix} 3 & 0 & 1 \\ -2 & 1 & 0 \\ -1 & 2 & 4 \end{pmatrix}$$

where $\alpha = \{(1,0,0), (0,1,0), (0,0,1)\}$

α is standard basis.

Theorem :

let A and B be similar matrices. Then

① A^2 and B^2 are also similar

② If A is invertible, so is B.

③ $(A - \lambda I)$ and $(B - \lambda I)$ are also similar,

for any $\lambda \in F$.

Q. Are $A = \begin{pmatrix} -2 & -1 \\ 5 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 2 \\ -1 & -1 \end{pmatrix}$?

similar ?

Soln.

Soln

Trace $A = 1$

Trace $B = 2$

Trace $A \neq$ trace B

we deduce that A and B are not

~~similiar~~ similar.

Q. ~~Are~~ let $A = \begin{pmatrix} 4 & 1 & -1 \\ 0 & 0 & 5 \\ 0 & 0 & -2 \end{pmatrix}$

$B = \begin{pmatrix} 0 & 1 & -2 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

Are A and B similar ?

Sol. No, $A - 4I = \begin{pmatrix} 0 & 1 & -1 \\ 0 & -4 & 5 \\ 0 & 0 & -6 \end{pmatrix}$

$B - 4I = \begin{pmatrix} -4 & 1 & -2 \\ 0 & -5 & 0 \\ 0 & 0 & -2 \end{pmatrix}$

$A - 4I$ is not invertible but $B - 4I$ is invertible

$\Rightarrow$ ~~A = 4A~~ $A - 4I$ is not invert.

$\Rightarrow$ $(A - 4I)$ and $(B - 4I)$ are not similar.

$\Rightarrow$ A and B are not similar.

Q. Let $A, B \in M_{n\times n}(\mathbb{R})$ be such that $AB = -BA$. If n is odd, then A or B is not invertible. True / False

Soln. True

$\det(AB) = \det(-BA)$

$\det(A)\det B = (-1)^n \det(B)\det(A)$

If n is odd, then

$\det(A)\det B = -\det(B)\det A$

$2\det(A)\det(B) = 0$

So either $\det(A) = 0$ or $\det B = 0$

$\Rightarrow$ A or B is not invertible.

Q. If x and y are vectors in an inner product space over $\mathbb{R}$ and $\|x+y\|^2 = \|x\|^2 + \|y\|^2$

then x and y ~~must~~ must be orthogonal.

True / False

Soln. True

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2$$

$$\Rightarrow \langle x+y, x+y\rangle - \langle x,x\rangle - \langle y,y\rangle = 0$$

$$\Rightarrow \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle - \langle x,x\rangle - \langle y,y\rangle = 0$$

$$\Rightarrow \langle x,y\rangle + \langle y,x\rangle = 0$$

$$\Rightarrow \langle x,y\rangle + \overline{\langle x,y\rangle} = 0$$

$$\Rightarrow 2\,\mathrm{Re}\langle x,y\rangle = 0$$

$$\Rightarrow \mathrm{Re}\langle x,y\rangle = 0$$

But this doesn't hold over $\mathbb{C}$

For $\mathbb{C}$.

let $\langle x,y \rangle = x\overline{y}$. Then $\langle x,y \rangle$ is an inner product on $\mathbb{C}$.

Take $x = 1, \; y = i$

$$\langle 1,i \rangle = 1(\overline{i})$$
$$= -i \neq 0$$

Also, $\|1+i\|^2 = \|1\|^2 + \|i\|^2$

since $Re\langle 1,i \rangle = Re(-i) = 0$

But x and y are not orthogonal.

Note :- $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Q. let $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the linear transformation defined by

$$T(a,b) = (2a-3b,\ a+b)$$

let $\alpha = \{(4,1), (-3,1)\}$. find ~~$[T]_{\mathbb{R}}$~~ $[T]_\alpha$.

Sol. let e be the standard ordered basis for $\mathbb{R}^2$.

$T(1,0) = (2,1)$

$T(0,1) = (-3,1)$

$\Rightarrow [T]_e = \begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix}$

$$[T]_\alpha = [I]_e^\alpha \; [T]_e \; [I]_\alpha^e$$

$$= \begin{pmatrix} 4 & -3 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 4 & -3 \\ 1 & 1 \end{pmatrix}$$

where $[I]_\alpha^e = \begin{pmatrix} 4 & -3 \\ 1 & 1 \end{pmatrix}$

$$= \frac{1}{7}\begin{pmatrix} 20 & -15 \\ 15 & 1 \end{pmatrix}$$

Q. let $A = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -6 & 9 \\ 0 & -1 & 0 \end{pmatrix}$.

find the Jordan Canonical form of A?

Sol.

Number of Jordan block J_i associated to λ = Nullity $(A-\lambda I)$

$Ch_A(x) = -(x+3)^3$

$x = -3$

$\Rightarrow$ Eigenvalues of A is -3.

$\Rightarrow$ Nullity $(A-\lambda I)$ = Nullity $(A+3I)$

$= \text{Nullity}\begin{pmatrix} 0 & 0 & 0 \\ 0 & -3 & 9 \\ 0 & -1 & 3 \end{pmatrix}$

$= 2$

Also, $m_A(x) = -(x+3)^2$

$\Rightarrow$ Number of Jordan block is 2.

Therefore Jordan canonical form of A

$$\text{is } J = \begin{pmatrix} \begin{pmatrix} -3 & 1 \\ 0 & -3 \end{pmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & (-3) \end{pmatrix}$$

$$= \begin{pmatrix} -3 & 1 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

Q. If two operators have the same characteristics polynomial, they must have the same Jordan canonical form. True / False

Sn. False

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Q. If a matrix A has Jordan canonical form J, then J^2 is the Jordan canonical form of A^2. True / False

Soln

$$J = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

then $J^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ but

J^2 is not the Jordan canonical form of any matrix.

Q. If a matrix A has Jordan canonical form J, then $-J$ is the Jordan canonical form of $-A$. True / False.

Soln. ~~B~~ False

By definition of Jordan block, it is an upper triangular matrix whose all diagonal entries are λ, all entries of the

Super diagonal (entries above the diagonal) are 1 and others are 0.

Take $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$-A = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$

Jordan canonical form of A and $-A$ is $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow -J$ is not Jordan canonical form of $-A$.

Q. Every operator on an ~~odd~~ odd dimensional space has a real eigenvalue. True / False

Sol. False

Consider $T : \mathbb{C} \longrightarrow \mathbb{C}$ with $T(x) = ix$

But this is true over $\mathbb{R}$.

Q. A vector space always contains infinitely many elements. True / False

Sol. False, $V = \{0\}$ is a vector space containing exactly one element.

Q. Let $\beta = \{(1,-1), (-1,2)\}$ be a basis of $\mathbb{R}^2$. Let $T \in L(\mathbb{R}^2)$ be the transformation that fixes v_1 and send v_2 to $-v_2$. Compute the matrix of T with respect to the standard basis of $\mathbb{R}^2$.

Sol^n

$$[T]_\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$[T]_\beta = [I]_e^\beta \, [T]_e \, [I]_\beta^e$$

$$[T]_e = [I]_\beta^e \, [T]_\beta \, [I]_e^\beta$$

where $[I]_\beta^e = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$ so that

$$[I]_e^\beta = \left([I]_\beta^e\right)^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

Thus,

$$[T]_e = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ -4 & -3 \end{pmatrix}$$

Projection along a vectors in $\mathbb{R}^n$:

Suppose $v \in \mathbb{R}^n$ is a vector. Then for $u \in \mathbb{R}^n$ define $\boxed{\text{proj}_v(u) = \frac{v \cdot u}{\|v\|^2} \cdot v}$

(1) Then $\text{proj}_v : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a linear transformation

since $\text{proj}_v(u+w) = \text{proj}_v(u) + \text{proj}_v(w)$

$\text{proj}_v(ku) = k\,(\text{proj}_v(u))$ where k is a scalar.

(2) The point of such projections is that any vector $u \in \mathbb{R}$ can be written uniquely as a sum of a vector along v and another one perpendicular to v.

$$\boxed{u = \text{proj}_v(u) + (u - \text{proj}_v(u))}$$

$$\boxed{(u - \text{proj}_v(u)) \perp \text{proj}_v(u)}$$

Q. Let $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the linear transformation that takes a point (x, y) and projects it orthogonally onto the line $y = -x$.

If β is the standard basis for $\mathbb{R}^2$. find $[T]_\beta$?

Sol. To orthogonally project the vector $\begin{pmatrix} x \\ y \end{pmatrix} = v$ onto the line $y = -x$, we first pick a direction vector for the line.

For instance $\vec{s} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

The orthogonal projection of $\vec{v}$ onto the line spanned by a non-zero $\vec{s}$ is

$$\text{proj}_{[\vec{s}]}(\vec{v}) = \frac{\vec{v} \cdot \vec{s}}{\vec{s} \cdot \vec{s}} \cdot \vec{s}$$

$$\Rightarrow T(x, y) = \text{proj}_s(v) = \frac{\begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}}{\begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix}} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= \frac{(x,y)\,(1,-1)}{\cancel{(1,1)}\,(1,-1)(1,-1)} \cdot (1,-1)$$

$$= \frac{(x-y)}{2} \cdot (1,-1)$$

$$= \left(\frac{x-y}{2} , \frac{-x+y}{2} \right)$$

$$T(x,y) = \left(\frac{x-y}{2} , \frac{-x+y}{2} \right)$$

$$T(1,0) = \left(\frac{1}{2} , -\frac{1}{2} \right)$$

$$T(0,1) = \left(-\tfrac{1}{2} , \tfrac{1}{2} \right)$$

$$[T]_\beta = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}$$

Note :- ① In $\mathbb{R}^3$, the orthogonal projection of a general vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ onto the y-axis is

$$\frac{(x,y,z)\,(0,1,0)}{(0,1,0)\,(0,1,0)} \cdot (0,1,0)$$

② The orthogonal projection of vector $\begin{pmatrix} x \\ y \end{pmatrix}$ onto the line $y = 2x$ is

$$\frac{\begin{pmatrix} x \\ y \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix}}{\begin{pmatrix} 1 \\ 2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix}} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= \frac{(x, y)\ (1, 2)}{(1, 2)\ (1, 2)} \cdot (1, 2)$$

Let v and u be two vectors in the inner product space V. Then orthogonal projection of u along v is

$$\boxed{P_v(u) = \frac{\langle u, v\rangle}{\|v\|^2} v}$$

Q. Let $V = M_n(\mathbb{R})$, $F = \mathbb{R}$ with inner product given by $\langle A, B\rangle = tr(AB^T)$.

Let W be the space of diagonal matrices. Find $W^\perp$?

Sol.

$$W^\perp = \{A \in M_n(\mathbb{R}) \mid \langle A, B\rangle = 0 \ \forall \ B \in W\}$$

$$= \{A \in M_n(\mathbb{R}) \mid tr(AB^T) = 0 \ \forall \ B \in W\}$$

It is given that W is the space of diagonal matrices so basis of W ~~so~~ basis is given by

$$B = \{e_{11}, e_{22}, e_{33}, \ldots, e_{nn}\}$$

$\Rightarrow W^{\perp} = \{ A \in M_n(\mathbb{R}) \mid \text{trace}(A e_{ii}^T) = 0$ for $i = 1, 2, \dots n \}$

$= \{ A \in M_n(\mathbb{R}) \mid a_{ii} = 0$ for $i = 1, 2, \dots n \}$

Therefore $W^{\perp}$ is collection of matrices having diagonal entries zero.

Q. Find the shortest distance of $(1, 1)$ from the line of $2y = x$?

Soln. Here $W = L(\{(2,1)\})$

$=$ linear span of $(2,1)$

$\|(2,1)\|^2 = 5$ so that orthonormal basis of W is $\{ \frac{(2,1)}{\sqrt{5}} \}$.

Thus, $P_W((1,1)) = \langle (1,1), (2,1) \rangle \frac{(2,1)}{5}$

$= \frac{3}{5} (2,1)$

by using the theorem

let $(V, \langle , \rangle)$ be an inner product space and W be its finite dimensional subspace. Then the shortest distance of a vector $v \in V$ is given by $\| v - P_W(v) \|$

$\Rightarrow$ the shortest distance of $(1, 2)$ from the line $y = 2x$ is $\| (1, 2) - \frac{3}{5}(2, 1) \|$

$$= \left\| \frac{(-1, 2)}{5} \right\|$$

$$= \frac{1}{\sqrt{5}}$$

Orthogonal Complement :

let A be a subset of an inner-product space H. The orthogonal complement of A is

$$A^{\perp} = \{x \in H : x \perp A\}$$
$$= \{x \in H : \langle x, y\rangle = 0 \text{ for all } y \in A\}$$

<u>Example</u>: let E be the set of all those vectors in ℓ^2 that have the form

$$x = (0, x_2, 0, x_4, \ldots)$$

and D be the set of all vectors in ℓ^2 of the form $x = (x_1, 0, x_3, 0 \ldots)$.

$$\boxed{D^{\perp} = E} \text{ and } \boxed{E^{\perp} = D}$$

The orthogonal complement of $\mathbb{R}^n$ is $\{0\}$,

* since the zero vector is the only vector that is orthogonal to all vectors in $\mathbb{R}^n$

$$\Rightarrow \{0\}^{\perp} = \mathbb{R}^n$$

(1) $U = \{(x, y, 0) \mid x, y \in \mathbb{R}\}$ is a 2D subspace of $\mathbb{R}^3$. The orthogonal Complement is $U^\perp = \{(0, 0, z) \mid z \in \mathbb{R}\}$

(2) $U = \{(x, ax, 0) \mid a, x \in \mathbb{R}\}$

$U^\perp = \{(-ay, y, z) \mid y, z \in \mathbb{R}\}$

(3) $U = \{0\}$ is a zero-dimensional subspace of $\mathbb{R}^3$.

$U^\perp = \mathbb{R}^3$.

Q. If V is a finite-dimensional inner product space and W is a subset of V, then $W = (W^\perp)^\perp$. True / False

False

Soln. This is true when W is a subspace but not generally for any subset.

For example:

$$W = \{(x,0) : x > 0\}$$

$$W^\perp = \{(0,y) : y \in \mathbb{R}\}$$

$$(W^\perp)^\perp = \{(x,0) : x \in \mathbb{R}\}$$

$$\Rightarrow W \neq (W^\perp)^\perp$$

Q. Let $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be defined by

$T(a_1, a_2) = (a_1 + a_2, 0, a_2)$.

Then T is one-one. True / False

Sol. True,

$T(a_1, a_2) = (a_1 + a_2, 0, a_2) = 0$

implies $a_1 + a_2 = a_2 = 0$.

The only solution is $a_1 = a_2 = 0$

$\Rightarrow$ $\text{Ker } T = \{0\}$

Therefore, T is one-one.

Q. Let $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be defined by

$T(a_1, a_2, a_3) = (a_1 + a_2, a_1, a_2)$. Then T

is one-one ? True / False

Sol. False.

$T(a_1, a_2, a_3) = (a_1 + a_2, a_1, a_2) = 0$

implies $a_1 + a_2 = a_1 = a_2 = 0$ and a_3 is

a free variable

Thus $\{(0,0,1)\}$ generates $\ker T$.

So $\ker T \neq \{0\}$ $\Rightarrow$ so T is not one-one.

Q. If A, B and C are $n \times n$ matrices, $A \neq 0$ and $AB = AC$, does it follow that $B = C$?

Sm. No,

$$A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$$

$$C = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}$$

$$AB = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$AC = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

But $B \neq C$

Q. The determinant of the following real matrix is given

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = 12.$$

find $\det\begin{pmatrix} d & e & 3f \\ a & b & 3c \\ g & h & 3i \end{pmatrix}$

Sol

$$\det\begin{pmatrix} d & e & 3f \\ a & b & 3c \\ g & h & 3i \end{pmatrix}$$

$$C_2 \longleftrightarrow \frac{C_3}{3}$$

$$= 3 \det\begin{pmatrix} d & e & \cancel{3}f \\ a & b & c \\ g & h & i \end{pmatrix}$$

$$r_1 \longleftrightarrow r_2$$

$$= -3 \det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = -3 \cdot 12$$

$$= -36$$

CRAMER'S RULE :

Consider the linear system

$$a_1 x + b_1 y = c_1$$
$$a_2 x + b_2 y = c_2$$

which in matrix format is

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Assume $a_1 b_2 - b_1 a_2$ non-zero

$$x = \frac{D_x}{D} = \frac{\det \begin{pmatrix} c_1 & b_1 \\ c_2 & b_2 \end{pmatrix}}{\det \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}}$$

$$= \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

$$y = \frac{D_y}{D} = \frac{\det\begin{pmatrix} a_1 & c_1 \\ a_2 & c_2 \end{pmatrix}}{\det\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}}$$

$$= \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

The rule for 3x3 matrices are similar.

Given

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

Which in matrix format is

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

$$x = \frac{D_x}{D} = \frac{\det\begin{pmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{pmatrix}}{\det\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}}$$

$$y = \frac{D_y}{D} = \frac{\det\begin{pmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{pmatrix}}{\det\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}}$$

$$= \frac{\begin{vmatrix} a_1 & \overset{d_1}{\cancel{b_1}} & \overset{c_1}{\cancel{d_1}} \\ a_2 & d_1\,\cancel{b_2} & \cancel{d_2}\,c_2 \\ a_3 & d_3\,\cancel{b_3} & \cancel{d_3}\,c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

$$z = \frac{D_z}{D} = \frac{\det\begin{pmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{pmatrix}}{\det\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}}$$

$$= \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

Q. Solve the following system over $\mathbb{R}$

$3x - 7y = -5$

$x + 10y = 4$

Sol.

$$x = \frac{\begin{vmatrix} -5 & -7 \\ 4 & 10 \end{vmatrix}}{\begin{vmatrix} 3 & -7 \\ 1 & 10 \end{vmatrix}} = \frac{-22}{7}$$

$$y = \frac{\begin{vmatrix} 3 & -5 \\ 1 & 4 \end{vmatrix}}{\begin{vmatrix} 3 & -7 \\ 1 & 10 \end{vmatrix}} = \frac{17}{37}$$