The lebesgue outer measure function.

$m^* : P(\mathbb{R}) \longrightarrow [0,\infty)$ defined by.

for all set $E \in P(\mathbb{R})$ by

$$m^*(E) = \inf\left\{ \sum_{n=1}^{\infty} l(I_n) : E \subseteq \bigcup_{n=1}^{\infty} I_n \right.$$

$$\left. \text{and } \{I_n = (a_n, b_n)\}_{n=1}^{\infty} \right\}.$$

Note: $P(\mathbb{R})$ mean subsets of $\mathbb{R}$, it can be finite or may be infinite.

Here $I_n = [a_n, b_n]$ or (a_n, b_n)

$m^*(E)$ is defined by taking all open interval or closed interval covers of E, ~~adda~~

$\Rightarrow$ after that adding all the length of open / closed interval in those cover and

and then taking the infimum of these values.

Example :- let $A \subset \mathbb{R}$ be finite ~~set~~ set.

Then $m^*(A) = 0$

proof:- let $A = (a_1, a_2, \ldots, a_n)$. Then

for every $\varepsilon > 0$ and for all $K \in \{1, 2, \ldots, n\}$

we have that $a_K \in \left(a_K - \frac{\varepsilon}{2}, a_K + \frac{\varepsilon}{2}\right)$,

~~So $A \subseteq \bigcup_{K=1}^{n} \left(a_K - \frac{\varepsilon}{2}, a_K + \frac{\varepsilon}{2}\right)$~~

So $A \subseteq \bigcup_{K=1}^{n} \left(a_K - \frac{\varepsilon}{2}, a_K + \frac{\varepsilon}{2}\right)$

we know that $\boxed{l(a, b) = b - a}$

Now the length corresponding to the open interval cover of A for each given ε is given by.

$$\sum_{K=1}^{n} l\left(\left(x_K - \frac{\varepsilon}{2}, x_K + \frac{\varepsilon}{2}\right)\right) =$$

$$= \sum_{k=1}^{n} \left(x_k - \frac{\varepsilon}{2} - x_k + \frac{\varepsilon}{2} \right)$$

Here $a = x_k - \frac{\varepsilon}{2}$, $b = x_k + \varepsilon/2$

$$= \sum_{K=1}^{n} \frac{2}{2} \times \varepsilon$$

$$= \sum_{k=1}^{n} \varepsilon = n\varepsilon$$

Therefore for all $\varepsilon > 0$, we have

$$m^*(A) < n\varepsilon$$

Now take $n = \underline{1}$, then $m^*(A) = 0$

$$m^*(A) = 0 < \varepsilon$$

where $\varepsilon = 1/2$.

$$0 < 1/2$$

Another Aproach / proof.

Theorem :- Every finite set has measure 0.

(step 1)

proof :- Let a finite set $A = \{x_i\}_{i=1}^{n}$

~~Sete~~ step 2 :- Take $I_i = [x_i - \frac{\varepsilon}{2n}, x_i + \frac{\varepsilon}{2n}]$

$\Rightarrow A \subset \cup_{i=1}^{n} I_i$.

Step 3 :- If we evaluate the ~~vat~~ value of $|I_i|$ (distance measure) for some $1 \le i \le n$, we have.

$$|I_i| = |[x_i - \frac{\varepsilon}{2n}, x_i + \frac{\varepsilon}{2n}]|$$

$$= x_i + \frac{\varepsilon}{2n} - (x_i - \frac{\varepsilon}{2n})$$

$$= \frac{2 \cdot \varepsilon}{2n} = \frac{\varepsilon}{n}.$$

Step 4 :- Now adding all n-interval, we see that $|\{x_i\}_{i=1}^{n}| \le \sum_{i=1}^{n} |I_i|$

$$= \sum_{i=1}^{n} \frac{\varepsilon}{n} = \frac{\varepsilon}{n} \sum_{i=1}^{n} 1.$$

$$= \frac{n\varepsilon}{n} = \varepsilon.$$

So, $m(A) < \varepsilon$.

implies $m(A) = 0$

Measure Subadditivity :

Let $(E_n)_{n=1}^{\infty}$ be a sequence of subsets of $\mathbb{R}$.

Then $\boxed{m^*\left(\bigcup_{n=1}^{\infty} E_n\right) \le \sum_{n=1}^{\infty} m^*(E_n)}$

proof: If we take $\sum_{n=1}^{\infty} m^*(E_n) = \infty$,

then obviously $m^*\left(\bigcup_{n=1}^{\infty} E_n\right) \le \sum_{n=1}^{\infty} m^*(E_n) = \infty$

i.e $m^*\left(\bigcup_{n=1}^{\infty} E_n\right) \le \sum_{n=1}^{\infty} m^*(E_n) = \infty$

It will be meaningless if ∞,

So, we assume that $\sum_{n=1}^{\infty} m^*(E_n)$ definite.

let $\varepsilon > 0$ and for every $n \in \mathbb{N}$, let

$(I_{n,m})_{m=1}^{\infty}$ be an open cover of E_n.

so $E_n \subseteq \bigcup_{m=1}^{\infty} I_{n,m}$.

we Know that

$$m^*(E_n) = \inf\left\{ \sum_{r=1}^{\infty} l(I_r) : E \subseteq \bigcup_{r=1}^{\infty} I_n \right.$$

$$\left. \text{and } \{ I_r = (a_r, b_r) \}_{r=1}^{\infty} \right\}$$

$(I_{n,m})_{m=1}^{\infty}$ is open interval cover of E_n

Now for $\varepsilon_0 = \dfrac{\varepsilon}{2^n} > 0$ we have

$$\sum_{m=1}^{\infty} l(I_{n,m}) < m^*(E_n) + \varepsilon_0$$

$$= m^*(E_n) + \frac{\varepsilon}{2^n}$$

Now since $\{I_{n,m}\}_{m=1}^{\infty}$ is an open interval cover of E_n, for each $n \in \mathbb{N}$, we have that $(I_{n,m})_{m,n=1}^{\infty}$ is an open interval cover of $\bigcup_{n=1}^{\infty} E_n$.

Therefore, $m^*\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} l(I_{n,m})\right)$

$$\leq \sum_{n=1}^{\infty}\left(m^*(E_n) + \frac{\varepsilon}{2^n}\right)$$

$$= \sum_{n=1}^{\infty} m^*(E_n) + \varepsilon \sum_{n=1}^{\infty} \frac{1}{2^n}$$

we have $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$

$$= \frac{1}{2}\left(\frac{1}{1-1/2}\right) = 1.$$

$$= \sum_{n=1}^{\infty} m^*(E_n) + \varepsilon$$

i.e $m^*\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} m^*(E_n) + \varepsilon$

Now for $\varepsilon > 0$, we have

$$\boxed{m^*\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} m^*(E_n)}$$

Hence proved

Theorem :- Every countable set has measure 0.

proof :- Let E be a countable subset of $\mathbb{R}$.

Let $x \in \mathbb{R}$, then $x \in [x-\varepsilon/2, x+\varepsilon/2]$

$$m(\{x\}) < m([x-\frac{\varepsilon}{2}, x+\frac{\varepsilon}{2}])$$

$$= m([x-\frac{\varepsilon}{2}, x+\frac{\varepsilon}{2}]) = |[x-\frac{\varepsilon}{2}, x+\frac{\varepsilon}{2}]|$$

$$= (x+\frac{\varepsilon}{2} - (x-\frac{\varepsilon}{2}))$$

$$= 2\frac{\varepsilon}{2} = \varepsilon.$$

Note: $\{x\} = [x, x]$

i.e. $m(\{x\}) = m\{[x,x]\} < \varepsilon$

$= 0 < \varepsilon \; \forall x \in \mathbb{R}.$

Now using the properties of subadditivity

we have $E = \bigcup_{n=1}^{\infty} \{x_n\}$ be countable

set

$\Rightarrow m(E) \le m(\bigcup_{n=1}^{\infty} \{x_n\}) \le \sum_{n=1}^{\infty} m(\{x_n\}) = 0$

i.e $\boxed{m(E) = 0}$

Length of open interval

The length $l(I)$ of an open interval I be defined by.

$$l(I) = \begin{cases} b-a & \text{if } I = (a,b) \text{ for some } a,b \in \mathbb{R} \text{ with } a<b \\ 0 & \text{if } I = \phi \\ \infty & \text{if } I = (-\infty, a) \text{ or } I = (a,\infty) \text{ for some } a \in \mathbb{R}. \\ \infty & \text{if } I = (-\infty, \infty) \end{cases}$$

Translation:-

If $t \in \mathbb{R}$ and $A \subset \mathbb{R}$, then the translation $t + A$ is defined by

$$t + A = \{t + a : a \in A\}$$

Outer Measure is translation invariant.

Suppose $t \in \mathbb{R}$ and $A \subset \mathbb{R}$,

Then $m^*(t + A) = m^*(A)$

step 1:

proof:- Let $(I_n)_{n=1}^{\infty}$ be a sequence of open interval that cover A.

Then $(I_n + t)_{n=1}^{\infty}$ is a sequence of open interval that cover $(t + A)$

Therefore $m^*(A + t) \leq \sum_{n=1}^{\infty} l(I_n + t)$

$$= \sum_{n=1}^{\infty} l(I_n) \quad \text{--- (1)}$$

Since the infimum of the last term over all sequence $I_1, I_2, I_3 \ldots\ldots$ of open interval whose union contain A.

Step 2 :- So for every sequence of open intervals $(I_n)_{n=1}^{\infty}$ that cover A we have that

$$m^*(A+t) \leq \sum_{n=1}^{\infty} l(I_n).$$

Hence $m^*(A+t) \leq m^*(A)$

Now let $(I_n)_{n=1}^{\infty}$ be a sequence of open intervals that cover $A+t$. Then $(I_n - t)_{n=1}^{\infty}$ is a sequence of open interval that cover A.

$$\therefore \; m^*(A) \leq \sum_{n=1}^{\infty} l(I_n - t) = \sum_{n=1}^{\infty} l(I_n)$$

Step 3 :- So for every sequence of open interval $(I_n)_{n=1}^{\infty}$ that cover $A+t$ we have that

$$m^*(A) \leq \sum_{n=1}^{\infty} l(I_n)$$

$$m^*(A) \leq m^*(A+t) \quad \text{—(2)}$$

From (1) and (2), we have

$$\boxed{m^*(A) = m^*(A+t)}$$

~~Open Cover~~ :

open cover :-

suppose $A \subset \mathbb{R}$, a collection C of open subsets of $\mathbb{R}$ is called an open cover of A if A is contained in the union of all the sets in C.

Example (1.) Take $\overset{\text{collection..}}{C} = \{(k, k+2) : k \in \mathbb{Z}^+\}$

is an open cover of $[2,4]$,

Here $[2,4] \subset \bigcup_{k=1}^{\infty} (k, k+2)$

$\downarrow$

this open cover has a finite subcover

because $[2,4] = [2,3] \cup [3,4]$

(2) Take C (collection) $= \{(0, 2 - 1/k) : k \in \mathbb{Z}^+\}$

is an open cover of $(1,2)$ because

$(1,2) \subset \bigcup_{k=1}^{\infty} (0, 2 - 1/k)$

Monotonicity in ~~Measure theory~~ Lebesgue (outer) measure.

if A and B are two set in $\mathbb{R}$ with $A \subset B$, then $\boxed{m^*(A) \leq m^*(B)}$

Q. Prove that if A and B are subset of $\mathbb{R}$ and $m^*(B) = 0$, then

$$m^*(A \cup B) = m^*(A)$$

proof: Since it given. $m^*(B) = 0$

so $B \subseteq A \cup B$

Now the property of monotonicity in outer measure

$$m^*(B) \leq m^*(A \cup B) \quad \text{--- (1)}$$

Now again the used the theorem. of Countable Subadditivity of the lebesgue of outer measure.

Theorem $\Rightarrow$ Let $(A_n)_{n=1}^{\infty}$ be a sequence of subsets of $\mathbb{R}$, Then $m^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} m^*(A_n)$

Note: Here union (symbol) changed to addition (summation).

i.e $m^*(A \cup B) \leq m^*(A) + m^*(B)$

$= 0 + m^*(B)$

$$\boxed{m^*(A \cup B) \leq m^*(B)} \quad - (2)$$

From (1) and (2), we have.

$$\boxed{m^*(A \cup B) = m^*(B)}$$

δ- Dilation

If we fix $\delta \geq 0$ and $A \subset \mathbb{R}$ and we define the δ - dilation of A as.

$$\boxed{\delta A = \{\delta a \mid a \in A\} \subseteq \mathbb{R}}$$

Then $m^*(\delta A) = \delta\, m^*(A)$

proof:- Fixed an interval $I = (a,b)$

Step 1 :- ~~Fix At~~ Then ~~$|I| = |b-a|$~~ $|I| = b-a$

and $|\delta I| = \delta(b-a)$

$\delta |I| = |\delta I|$

$\Rightarrow \sum \delta |I_n| = \sum |\delta I_n|$ where $\llcorner$ (1)

I_n is interval.

Step 2 :- Let $\{I_n\}$ cover A. Then we can dilate ~~each~~ each interval by δ.

Let $\{\delta I_n\}$ be the dilated collections of intervals

Step ③ :- Fix $x \in (a, b)$. Then $\delta x \in (\delta a, \delta b)$.

So we have ~~A~~: $\delta A \subset \cup \{\delta I_n\}$

Since $\{I_n\}$ cover A

Now ~~ut~~ using the properties of subadditivity

$$m\cancel{^*(\delta A) \leq m^*(\cup \delta I_n}$$

$$m^*(\delta A) \leq m^*(\cup\{\delta I_n\})$$
$$\leq \sum |\delta I_n|$$

i.e $\boxed{m^*(\delta A) \leq \sum |\delta I_n|}$

~~From~~ Step ④ : From ①, we have:

$$\sum \delta |I_n| = \sum |\delta I_n|$$

so $m^*(\delta A) \leq \sum \delta |I_n|$

Step 5 :- By the definition of the infimum, for all $\varepsilon > 0$, we can find a cover of A called $\{I_n\}$

Such that $\sum |I_n| < m^*(A) + \varepsilon$.

This implies $\sum \delta |I_n| < \delta m^*(A) + \delta\varepsilon$

Therefore, we can conclude

$$\boxed{m^*(\delta A) < \delta m^*(A) + \delta \varepsilon}$$

Step 6 :- let $\varepsilon \longrightarrow 0$, then we have

$$\boxed{m^*(\delta A) \leq \delta m^*(A)} \quad \text{(1)}$$

which hold for all $A \subset R$ and $\delta \geq 0$.

Step 7 :- let $\delta = 0$, Then $\delta A = \{0\}$

Since $\delta m^*(A) = 0 \cdot m^*(A) = 0 = m^*(0)$

Step 8 : let $\delta \neq 0$. Then define $B = \delta A$

and $\lambda = \delta^{-1}$,

Then we have $\boxed{m^*(\lambda B) \leq \lambda m^*(B)}$

But $\lambda B = A$

since $B = \delta A$.

$$\delta = \frac{1}{\lambda} \Rightarrow \boxed{\lambda = \delta^{-1}}$$

$$\Rightarrow \boxed{A = \lambda B}$$

So we have $m^*(A) \leq \delta^{-1} m^*(\delta A)$

$$\boxed{\delta\, m^*(A) \leq m^*(\delta A)}$$

<u>Hence proved</u>

$\Rightarrow$ From step 6 and 7, we have.

$$\boxed{m^*(\delta A) = \delta\, m^*(A)}$$

Q. Prove that $m^*([0,1] - \mathbb{Q}) = 1$.

proof:- Step 1:

Consider that $[0,1] - \mathbb{Q} \subset [0,1]$

Thus $m^*([0,1] - \mathbb{Q}) \leq 1$. — (1)

Since $m^*([0,1]) = l([0,1]) = 1$.

Step 2: Consider the $A = [0,1] - \mathbb{Q}$

$B = [0,1] \cap \mathbb{Q}$.

$A \cup B = [0,1]$

~~[0,1] - Q [0]~~

$$\boxed{[0,1] - \mathbb{Q} \cup [0,1] \cap \mathbb{Q} = [0,1]}$$

$$\boxed{m^*(A \cup B) \leq m^*(A) + m^*(B)}$$

↓

Countable subadditivity of the lebesgue of outer measure

$$1 = m^*([0,1]) \leq m^*([0,1] - \mathbb{Q}) + m^*([0,1] \cap \mathbb{Q})$$

Since $[0,1] \cap \mathbb{Q} \subset \mathbb{Q}$,

$$\Rightarrow m^*([0,1] \cap \mathbb{Q}) \leq 0$$

$$\Rightarrow m^*([0,1] \cap \mathbb{Q}) = 0$$

So we have $1 \leq m^*([0,1] - \mathbb{Q})$ — (2)

from (1) and (2)

$$\boxed{m^*([0,1] - \mathbb{Q}) = 1}$$

Q let $A = \bigcup_{i=1}^{\infty} A_i$ and $A_1 \subseteq A_2 \subseteq A_3 \ldots$

is an increasing sequence where $A_i \subseteq [0,1]$

prove that $m^*(A_n) \longrightarrow m^*(A)$

proof:- $A_1 \subseteq A_2 \subseteq A_3 \ldots\ldots$

$B_i = A_i \setminus A_{i-1}$ $\forall\ i = 1$ to ∞

$B_1 = A_1$.

$B_2 = A_2 - A_1$

$B_3 = A_3 - A_2$

$\vdots$

$B_i = A_i - A_{i-1}$.

Here B_i are disjoint pairwise.

$\cancel{\bigcup_{i=1}^{n} B}$ $\cancel{\bigcup_{i=1}^{\infty}}$ $\bigcup_{i=1}^{n} B_i = \bigcup_{1}^{n} A_i$

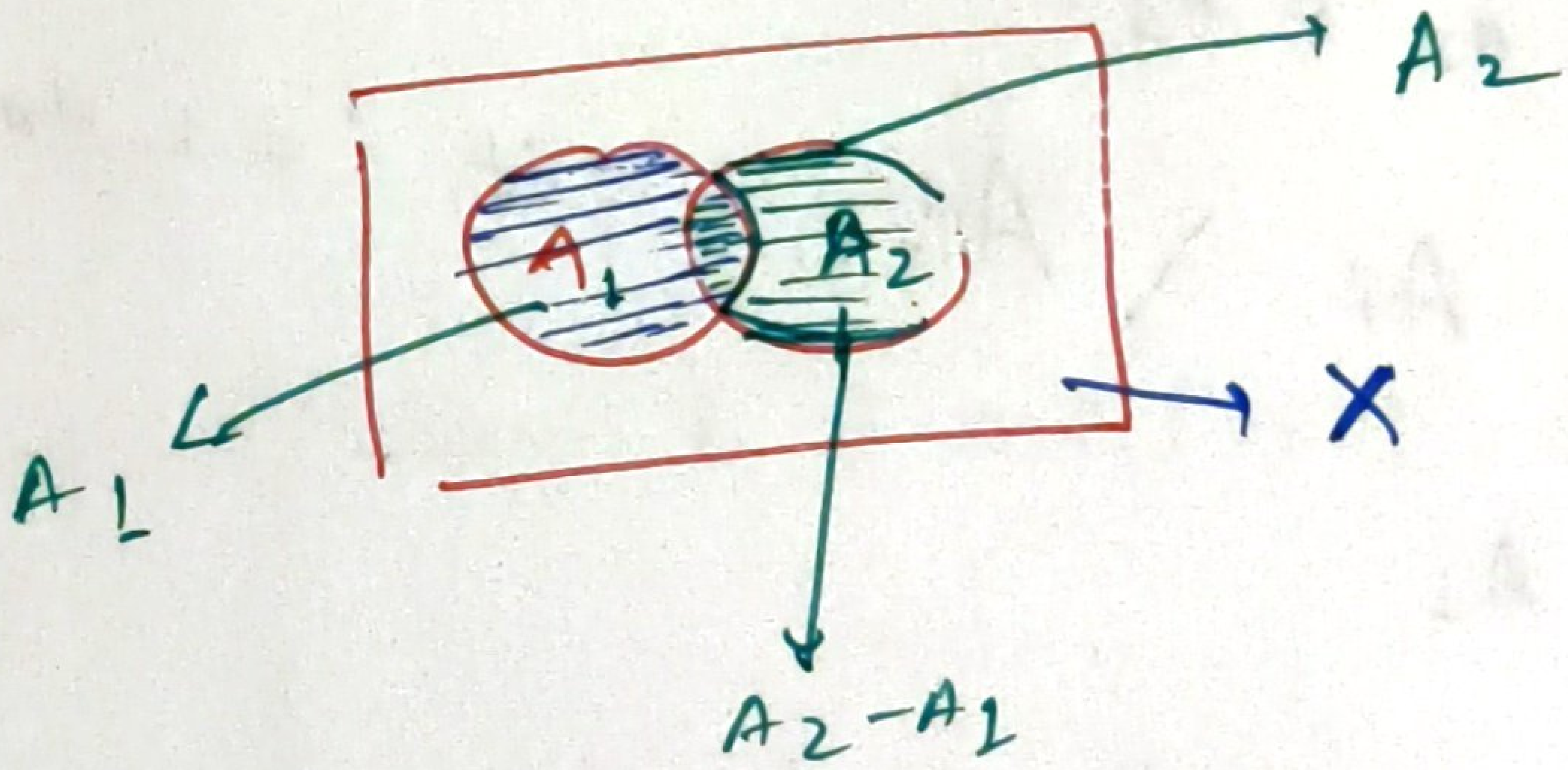
$\Rightarrow A_1 \cup (A_2 - A_1) \cup (A_3 - A_2) \ldots\ldots \cup (A_n - A_{n-1})$

$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots\ldots \cup A_n$.

we can check at $n=2$,

$$A_1 \cup (A_2 - A_1) = A_1 \cup A_2$$

By using venn diagram



$$A_1 \cup (A_2 - A_1) = A_1 \cup A_2$$

De-Morgan law:

$$A_1 \cup (A_2 - A_1) = A_1 \cup (A_2 \cap A_1^c)$$

$$= (A_1 \cup A_2) \cap (A_1 \cup A_1^c)$$

$$= (A_1 \cup A_2) \cap (X)$$

$$\boxed{A_1 \cup (A_2 - A_1) = A_1 \cup A_2}$$

$$\boxed{A_1 \cup (A_2 - A_1) = A_1 \cup A_2}$$

similarly $\boxed{\bigcup_{i=1}^{n} B_i = \bigcup_{i=1}^{n} A_i}$ for all ~~set~~ n.

Therefore $\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} A_i$

$$m^*(A) = m^*\left(\bigcup_{i=1}^{\infty} A_i\right)$$

$$= m^*\left(\bigcup_{i=1}^{\infty} B_i\right)$$

Now use the theorem of countable additivity of outer measure.

$$= \sum_{i=1}^{\infty} m^*(B_i)$$

$$= \lim_{n\to\infty} m^*\left(\bigcup_{i=1}^{n} B_i\right)$$

$$= \lim_{n\to\infty} m^*\left(\bigcup_{i=1}^{n} A_i\right) \qquad \left\{\because \bigcup_{i=1}^{n} A_i = A_n\right\}.$$

$$= \lim_{n\to\infty} m^*(A_n) = m^*(A)$$

Hence proved

σ- Algebra (Sigma Algebra) : (Measurable space)

let X be any set and let F be a collection of subsets of X.

⇒ A σ-Algebra F of subsets of X is a collection F of subset of X satisfying the following conditions:

(1) $\phi \in F$

(2) if $B \in F$ then B^c is also in F

i.e if $B \in F$, then $B^c \in F$

(3) if $B_1, B_2, B_3 \dots$ is a countable collection of sets in F then their union $\bigcup_{n=1}^{\infty} B_n \in F$

i.e if ~~A~~ $B_1, B_2 \dots \in F$, then $\bigcup_{n=1}^{\infty} B_n \in F$

Note: Σ = Sigma (~~Syntax~~ symbolic)

$\boxed{\Sigma = F}$ ⇒ collection.

Example :-

Now we can say ~~to~~ that

(X, F) or (X, Σ) is a measurable

space.

Example :- Take ~~F = {ϕ,~~ ~~{~~

$$X = \{1, 2, 3, 4\}.$$

Then $F = \{\phi, \{1, 2\}, \{3, 4\}, \{1, 2, 3, 4\}\}.$

Take $\ni$ $B = \{1, 2\}$

$B^c = \{3, 4\}.$

i,e $F = \{\phi, B, B^c, X\}.$

Q. Show that the characteristic function of a set E is measurable if and only if E is measurable.

Soln: Suppose that E is measurable space X.

Then

$$\cancel{\chi_E(x) = \begin{cases} 1 & \text{if } x \in \\ & \end{cases}}$$

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

Let $\alpha \in \mathbb{R}$, then.

① $\{x : \chi_E(x) < \alpha\} = \phi$ if $\alpha \leq 0$

② $\{x : \chi_E(x) < \alpha\} = X$ if $\alpha > 1$.

i.e $\{x : \chi_E(x) < \alpha > 1\} = X$

③ $\{ X : \chi_E(x) < \alpha$ where $\alpha \in (0,1] \}$

$$= E^c \quad \text{if } 0 < \alpha \leq 1$$

$$= X - E$$

④ $\{ X : \chi_E(x) < \alpha$ where $\alpha \in [0,1) \}$

$$= E \quad \text{if } 0 \leq \alpha < 1$$

In all case $\{x : \chi_E(x) < \alpha\}$

$$= \chi_E^{-1}(-\infty, \alpha)$$

$\Rightarrow \chi_E^{-1}(-\infty, \alpha)$ is measurble.

$\Rightarrow \chi_E$ is measurable.

$$\boxed{\chi_E^{-1}(-\infty, \alpha) = \{x : \chi_E(x) < \alpha\}}$$

$$\chi_E^{-1}(-\infty, \alpha) = \begin{cases} X & \text{if } \alpha > 1 \\ E^c & \text{if } 0 \leq \alpha < 1. \\ E & \text{if } 0 < \alpha \leq 1. \\ \phi & \text{if } \alpha \leq 0. \end{cases}$$

$$\mathcal{F} = \{\phi, E, E^c, X\}.$$

so $(X, \mathcal{F})$ is a measurable space.

Note:- $\chi_E(x) < 1/2$ if $\chi_E(x) = 0$

$$X - E = \{x : \chi_E(x) < 1/2\}$$
$$= \{x : \chi_E(x) = 0\}$$

$$E = X - (X - E)$$
$$= \{x : \chi_E(x) \geq 1/2\}.$$
$$= \{x : \chi_E(x) = 1\}.$$

Theorem : Let X, Y, Z be topological spaces such that

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

① If f and g are continuous, then $g \circ f$ is continuous

proof:- $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$

$= f^{-1}(\text{open})$

$= \text{open}.$

Note:- If X and Y are topological space and if f is a mapping of X into Y, then f is said to be continuous if $f^{-1}(V)$ is an open set in X for every open set V in Y.

Theorem :- let $u : X \longrightarrow R$, $v : X \longrightarrow R$

and $\phi : \mathbb{R} \times \mathbb{R} \longrightarrow Y$.

i.e let u and v be real measurable functions on a measurable space X. let ϕ be a contonious mapping of the plane into a topological space Y ~~and~~ defined by.

$$h(x) = \phi(u(x), v(x)) \quad \text{for } x \in X.$$

Then $h : X \longrightarrow Y$ is measurable.

proof:- To prove this theorem, we must know one theorem

Theorem:- let S, T and R be metric space.

let $f : S \longrightarrow T$ and $g : T \longrightarrow R$,

now define the composition function

$g \circ f : S \longrightarrow R$ by

$$g \circ f(s) = g(f(s))$$

① If $U \subset R$, then $(g \circ f)^{-1}(U)$

$= f^{-1}(g^{-1}(U))$

Sol:

Let $x \in (g \circ f)^{-1}(U)$.

Then $g \circ f(x) \in U$.

$\Rightarrow g(f(x)) \in U$.

$\Rightarrow f(x) \in g^{-1}(U)$.

$\Rightarrow x \in f^{-1}(g^{-1}(U))$

Therefore $\boxed{(g \circ f)^{-1}(U) \subset f^{-1}(g^{-1}(U))}$

② If f and g are continuous, then the composition $g \circ f$ is also continuous.

Proof: Let U be open in R. Since $U \subset R$ and g is continuous $g : T \longrightarrow R$.

$\Rightarrow g^{-1}(U) \subset T$ is open.

Now for f

Here f is continuous $f: S \longrightarrow T$

~~for~~ so $f^{-1}(g^{-1}(U)) \subset S$

since $g^{-1}(U) \subset T$

$\Rightarrow f^{-1}(g^{-1}(U))$ is open.

From (1) we have show that $(g \circ f)^{-1}$ is open., so $g \circ f$ is continuous.

Theorem:- Let Y and Z be topological spaces and let $g: Y \longrightarrow Z$ be continuous.

(a) If X is a topological space. If $f: X \longrightarrow Y$ is continuous, and if $h = g \circ f$, then $h: X \longrightarrow Z$ is continuous

(b) If X is a measurable space. If $f: X \longrightarrow Y$ is measurable, and if $h = g \circ f$, then $h: X \longrightarrow Z$ is measurable.

proof:- (1) if V is open in Z, then

$g^{-1}(V)$ is open in Y.

we know that $h = g \circ f$

$(h)^{-1} = g \circ f\,(V)$

$\Rightarrow (h)^{-1} = (g \circ f)^{-1}(V)$.

$\Rightarrow h^{-1} = f^{-1}((g^{-1}(V)))$

i.e f^{-1} is open since f is

continuous

Therefore $h^{-1}(V)$ is also open

this show that $h : X \rightarrow Z$ is

continuous.

$\Rightarrow$ if f is measurable, $\therefore$ then f^{-1} is open.

implies $h^{-1}(V)$.

To show that if $R \subseteq \mathbb{R}^2$ is an open rectangle, then $f^{-1}(R) \in M$.

$\Rightarrow$ let $(a,b), (c,d) \subseteq R$ be open interval such that $R = \{(y,z) \in \mathbb{R}^2 \mid a<y<b,\ c<z<d\}$.

if $(u(x), v(x)) \in R$, then $u(x) \in (a,b)$
$v(x) \in (c,d)$

$\Rightarrow$ $u(x) \in (a,b)$ ~~&~~ $v(x) \in (c,d)$

this implies $\boxed{x \in u^{-1}(a,b) \cap v^{-1}(c,d)}$ — ①

Since we have already define

$$f : X \longrightarrow \mathbb{R}^2 \text{ by}$$

$$f(x) = (u(x), v(x))$$

$$\Rightarrow x = f^{-1}(u(x), v(x))$$

$$x = u^{-1}(a,b) \cap v^{-1}(c,d)$$

since $x \in u^{-1}(a,b) \cap v^{-1}(c,d)$ form ①

$\Rightarrow$ ~~we know $(u(x), v(x)) \in R$.~~

$(u(x), v(x)) \in R$

and $(a,b), (c,d) \subseteq \mathbb{R}$

$f^{-1}(R) = u^{-1}(a,b) \cap v^{-1}(c,d).$

Since u is measurable, $u^{-1}(a,b) \in \mathcal{M}$

Since v is measurable, $v^{-1}(c,d) \in \mathcal{M}$

$\mathcal{M}$ is σ - algebra

Therefore $u^{-1}(a,b) \cap v^{-1}(c,d) \in \mathcal{M}$.

and $f^{-1}(R) \in \mathcal{M}$ for any rectangle R.

Since $\mathcal{M}$ is closed under countable intersections.

Now take any open set $V = \bigcup_{i=1}^{\infty} R_i$

where V is rectangle around points with rational coordinates,

So $f^{-1}(V) = f^{-1}\left(\bigcup_{i=1}^{\infty} R_i\right)$

$= \bigcup_{i=1}^{\infty} f^{-1}(R_i)$

Each term in the union is in M,

so since countable unions of elements in M are in M

$f^{-1}(V) \in M$.

i.e $f^{-1}(V)$ is measurable.

Hence proved.

Theorem :- If $\mathcal{B}$ a non-empty collection of subsets of X, there exist a smallest σ-algebra containing ~~all the~~ all the sets of $\mathcal{B}$ is denoted as $\sigma(\mathcal{B})$

$\sigma(\mathcal{B})$ = Sigma ~~at~~ algebra ~~do~~ generated by the collection β.

In Rudin book: $\mathcal{M}^*$ $\rightarrow$ σ - algebra ~~generat~~ generated by $\mathcal{F}$

where $\mathcal{F}$ is any collection of subsets of X.

Borel set :-

~~Let X be topological space, then the family of Borel set in X is the σ- of~~

Let X be topological space, ~~then~~ then the ~~σfa~~ of family of borel set is the σ - algebra generated by the family of opensets

i.e there exist a smallest σ - algebra containing all the set B in X such that every open set in X belong to B. The member of B are called Borel set.

Example :- $A = [a, b]$, A is closed in $\mathbb{R}$.

$A^c = (-\infty, a) \cup (b, \infty)$ is open in $\mathbb{R}$.

By property sigma algebra, :- if B in $\mathbb{F}$, then B^c also in $\mathbb{F}$

$\Rightarrow$ Borel sigma contains all open set as it contain A^c,

$$A^c = (-\infty, a) \cup (b, \infty)$$

$$(A^c)^c = [a, b] = A$$

$$(A^c)^c = A.$$

<u>Notes</u>: The collection of Borel subset of $\mathbb{R}$, is the smallest collection of subsets of $\mathbb{R}$ closed under taking countable union,

The characteristic function of a subset A of a set X is a function

$$\chi_A : X \longrightarrow \{0, 1\} \text{ defined.}$$

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

~~let A B~~

let $B \subset X$

Then ① $\boxed{\chi_{A \cap B} = \chi_A \cdot \chi_B.}$

Proof: If $x \in A \cap B$, then

$$\chi_A(x) = \chi_B(x) = 1.$$

$$\Rightarrow \chi_{A \cap B}(x) = 1$$

Here $x \in A$, $x \in B$

so $\chi_A(x) \cdot \chi_B(x) = 1 \cdot 1 = 1.$

if $x \notin A \cap B$, then $\chi_A(x) = 0$, $\chi_B(x) = 0$

$$\Rightarrow \chi_{A \cap B}(x) = 0$$

again $\chi_A \cdot \chi_B = 0 \cdot 0 = 0$

so $\boxed{\chi_{A \cap B} = \chi_A \cdot \chi_B}$

(2) $\boxed{\chi_{A \cup B} = \chi_A + \chi_B - \chi_{A \cap B}}$

if $x \in A \cup B$, then $\chi_{A \cup B} = 1$.

$x \in A$, then $\chi_A = 1$

$x \in B$, then $\chi_B = 1$. } from (1)

$x \in A \cap B$, then $\chi_{A \cap B} = 1$

$\chi_{A \cup B} = 1 + 1 - 1 = 1$.

Similarly if $x \notin A \cup B$, then $\chi_{A \cup B} = 0$

$x \notin A$, then $\chi_A = 0$

$x \notin B$, then $\chi_B = 0$

$x \notin A \cap B$, then $\chi_{A \cap B} = 0$

$\chi_{A \cup B} = 0 + 0 - 0 = 0$.

Theorem :

If $f_n : X \longrightarrow [-\infty, \infty]$ is measurable,

for $n = 1, 2, 3, \ldots$ and

$$g = \sup_{n \geq 1} f_n$$

$$h = \limsup_{n \to \infty} f_n$$

then g and are measurable.

proof:- If ~~$f_n(x) >$~~ $f_n(x) > \alpha$ for some $n \in \mathbb{N}$.

then $g(x)$ is an upper bound of the set

$\{f_n(x) \mid n \in \mathbb{N}\}$ such that

$$\boxed{g(x) \geq f_n(x) > \alpha}$$

i,e $\quad g(x) \in (\alpha, \infty]$

$x \in g^{-1}((\alpha, \infty])$ implies

$g(x) > \alpha$.

if $g(x) > \alpha$, then there exist $n \in \mathbb{N}$

such that $f_n(x) > \alpha$.

So $x \in \bigcup_{n=1}^{\infty} f_n^{-1}((\alpha, \infty])$

To prove this ~~the~~ theorem, we must know this theorem also,

Theorem:- Suppose that (X, A) and $(Y, \mathcal{B})$ are measurable space and $\mathcal{B} = \sigma(\mathcal{G})$ is generated by a family $\mathcal{G} \subset P(Y)$ (power set of Y)

Then $f: X \longrightarrow Y$ is measurable if and only if $f^{-1}(G) \in A$ for every $G \in \mathcal{G}$

proof:- we $f^{-1}(Y) = X$

$$f^{-1}(Y - B) = f^{-1}(Y) - f^{-1}(B)$$

$$f^{-1}(Y - B) = X - f^{-1}(B)$$

$$f^{-1}\left(\bigcup_{i=1}^{\infty} B_i\right) = \bigcup_{i=1}^{\infty} f^{-1}(B_i)$$

$$f^{-1}\left(\bigcap_{i=1}^{\infty} B_i\right) = \bigcap_{i=1}^{\infty} f^{-1}(B_i).$$

It follows that

$$M = \{B \subset Y : f^{-1}(B) \in A\} \text{ is a}$$

σ-algebra on Y.

M contain $\mathcal{G}$ so $\mathcal{G}$. $\mathcal{G} \subset M$

Therefore $\sigma(\mathcal{G}) = \mathcal{B} \subset M$

which implies that f is ~~measurable~~. measurable

Therrem :- If $f = g - h$, $g \geq 0$ and $h \geq 0$

then $f^{+} \leq g$ and $f^{-1} \leq h$

proof:- $f \leq g$ and $g \geq 0$ clearly implies

$$f^{+} = \max\{f, 0\} \leq g.$$

$$-f \leq 0 \text{ and } +\cancel{f} \leq \cancel{-h}.$$

$$f^{-} = -\min\{f, 0\} = -\min\{-h, 0\}$$

$$\leq -(-h)$$

$$\boxed{f^{-} \leq h.}$$

Simple Function :-

A complex function on a measurable space X whose range consists of only finitely many points is called a simple function.

$\Rightarrow$ A simple function is a finite linear combination of indicator function of measurable sets.

Let (X, Σ) be a measurable space.

Let $A_1, A_2 \ldots A_n \in \Sigma$ be sequen Sequence of disjoint measurable set

i.e $A_i \cap A_j = \phi$ if $i \neq j$

$$X = A_1 \cup A_2 \ldots \cup A_n.$$

Let $a_1, a_2 \ldots a_n$ be a sequence of real or complex numbers.

A simple function $f : X \longrightarrow \mathbb{C}$ of the form

$$\boxed{f(x) = \sum_{k=1}^{n} a_k \mathbb{1}_{A_k}(x)}$$

Take $V \subseteq \mathbb{C}$,

Then f is measurable since

$$f^{-1}(V) = \bigcup \{A_i : a_i \in V\}. \text{ is}$$

measurable.

Take $V = V_i = (a_i - \varepsilon, a_i + \varepsilon)$

$$f = \sum_{i=1}^{n} a_i X_{A_i}$$

$$X_{A_i} = \begin{cases} 1 & \text{if } \cancel{A_i}\ x \in A_i \\ 0 & \text{if } x \notin A_i \end{cases}$$

$$f = \sum_{i=1}^{n} a_i \quad \text{if} \quad x \in A_i.$$

$f(x) = a_i$ implies $f(x) \in V_i$

~~since~~ ~~A_i~~, so $f(A_i) = V_i$

so $A_i = f^{-1}(V_i)$

$= f^{-1}(V)$ is measurable subset of

$\mathbb{R}$.

Example :-

① Every Constant function defined on a lebesgue measurable set is a simple function

Such as $f : \mathbb{R} \longrightarrow \{k\}$ for all $x \in \mathbb{R}$

by $f(x) = k$ ~~where~~ where $k \in \mathbb{R}$

② let $A \subseteq B$. The characteristic function of A on the set B is the function

$\chi_A : B \longrightarrow \{0, 1\}$ defined for all

$x \in B$ by.

$$\chi_A(x) = \begin{cases} 0 & \text{if } x \notin A \\ 1 & \text{if } x \in A. \end{cases}$$

χ_A is simple function since B is a lebesgue measurable set due ~~funct~~ to function range is finite.

Theorem :-

let $f : X \longrightarrow [0, \infty]$ be measurable.

There exist simple ~~m~~ measurable function S_n on X such that

(a) $0 \leq S_1 \leq S_2 \leq \ldots \leq f$

(b) $S_n(x) \longrightarrow f(x)$ as $n \longrightarrow \infty$ every $x \in X$.

proof :-

Step 1: Here we will used the

Archimedean property :

i.e if $\frac{t}{\delta_n}$ is any real number such that

$\frac{t}{\delta_n} > 0$, then there exist a natural number $K+1 \in \mathbb{N}$

such that $K \leq \frac{t}{\delta_n} < (K+1)$

$$K \le \frac{t}{\delta_n} < (K+1)$$

$$\boxed{K\delta_n \le t < \delta_n (K+1)} \quad (1)$$

Now Take $\delta_n = \frac{1}{2^n}$

$$K = K_n(t)$$

Now define simple measurable function

$$S_n(t) = \sum_{K=0}^{n.2^n-1} K\delta_n \, \chi_{[K\delta_n, (K+1)\delta_n)}$$

$$+ \sum_{K=0}^{n.2^n-1} n \, \chi_{[n, \infty)}$$

$$\boxed{S_n(t) = \sum_{K=0}^{n.2^n-1} K \cdot \delta_n \, \chi_{[K\delta_n, (K+1)\delta_n)} + n \, \chi_{[n,\infty)}}$$