Q. There is no function $f: \mathbb{R} \longrightarrow \mathbb{R}$ which is continuous on $\mathbb{Q}$ and discontinuous on $\mathbb{R} - \mathbb{Q}$. ~~True~~ True/False

Soln. True, because the points of continuity of a function form a G_δ set

where $G_\delta = \bigcap_{n \in \mathbb{N}} U_n$ i.e. the intersection of a countable collection of open sets.

Let $x \in U_n$ and for some $\varepsilon > 0$ such that for all y and $z \in (x-\varepsilon, x+\varepsilon)$,

then $|f(y) - f(z)| < \frac{1}{n}$

$\Rightarrow f$ is continuous

Now using baire category theorem, the rational number are not a G_δ.

Therefore, there is no function $f: \mathbb{R} \longrightarrow \mathbb{R}$ which is continuous on $\mathbb{Q}$ or all points of $\mathbb{Q}$.

There exist a function $f: \mathbb{R} \to \mathbb{R}$ which is continuous on single points of $\mathbb{Q}$ and discontinuous on $\mathbb{R}-\mathbb{Q}$. True/False

Sol. True

Take $f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ -x & \text{if } x \text{ is irrational.} \end{cases}$

Q. let A be component of X and let $f: X \longrightarrow Y$ be continuous. Show that through a counter examples that f(A) need not be a component of the subspace f(x) of Y.

Soln:

Note: A Component C of a topological space X is a maximal connected subspace.

Here, Take $X = [0,1] \cup [2,3]$, $Y = \mathbb{R}$

Define $f: X \longrightarrow Y$ defined by

$$f(x) = \begin{cases} x & \text{if } x \in [0,1] \\ x-1 & \text{if } x \in [2,3] \end{cases}$$

$\Rightarrow$ f is continuous, but the image of the component [0,1] of X is not component of $f(X) = [0,2]$

Q. If $\sum_{n=1}^{\infty} \sqrt{a_n a_{n-1}}$ is convergent then

$\sum a_n$ is convergent ? True/False

Soln. False

$$a_n = \begin{cases} n^2 & \text{if } n \text{ is even} \\ \frac{1}{n^{100}} & \text{if } n \text{ is odd} \end{cases}$$

$\sum a_n$ is divergent but each term

of $\sqrt{a_n a_{n-1}} \approx \frac{1}{n^{49}}$

$\Rightarrow \sum_{n=1}^{\infty} \sqrt{a_n a_{n-1}}$ is convergent.

Q. Let $a_n = \left(\frac{1+(-1)^n}{2^n}\right) + \left(\frac{1+(-1)^{n-1}}{3^n}\right)$.

Then find the radius of convergence of the following power series $\sum_{n=1}^{\infty} a_n x^n$ about $x=0$?

Sol^n.

$$a_{2n} = \frac{1}{2^{n-1}} = \frac{2}{2^n}$$

$$\text{Ily } a_{2n+1} = \frac{2}{3^n}$$

$$R_1 = \frac{1}{\lim\limits_{n\to\infty} \sup |a_{2n}|^{1/n}} = 2$$

$$R_2 = 3 = \frac{1}{\lim\limits_{n\to\infty} \sup |a_{2n+1}|^{1/n}} = 3$$

Radius of Convergence $= \min(R_1, R_2)$

$= 2$

Q. Determine the set of all points where the Taylor series of the function

$$f(x) = \sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n}$$ around the point $x = e$ converge to $f(x)$.

Soln. $f(x) = \sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n} = x^2 \sum_{n=0}^{\infty} \frac{1}{(1+x^2)^n}$

$$= x^2 \cdot \frac{1}{1 - \frac{1}{(1+x^2)}}$$

$f(x) = 1 + x^2$ for $\left| \frac{1}{1+x^2} \right| < 1$ i.e. $x \neq 0$

Also, $f(0) = 0$

$f(e) = 1 + e^2$

$f'(e) = 2e$

$f''(e) = 2$

$\vdots$

$f^n(e) = 0$ for $n \geq 3$

Taylor series of f about $x = e$

$$= 1 + e^2 + \frac{2e(x-e)}{1!} + \frac{(x-e)^2}{2!}$$

It converges to $f(x)$ except when $x = 0$ or

i.e $\mathbb{C} - \{0\}$ or $\mathbb{R} - \{0\}$

Since f is an entire function.

therefore it converge to $f(x)$ when $x \in \mathbb{C} - \{0\}$

Note:- $$\sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n} = \begin{cases} 1+x^2 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Taylor series: $$\sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$

Q. For any positive real numbers α and β, define

$$f(x) = \begin{cases} x^\alpha \sin\left(\frac{1}{x^\beta}\right), & \text{if } x \in (0,1] \\ 0 & \text{if } x = 0 \end{cases}$$

(a) For a given $\beta > 0$, find all values of α such that $f'(0)$ exists.

(b) For a given $\beta > 0$, find all values of α such that f is of bounded variation on $[0,1]$.

Soln. (a) $$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{x \to 0} \frac{x^\alpha \sin \frac{1}{x^\beta}}{x}$$

$$= \lim_{x \to 0} x^{\alpha - 1} \sin \frac{1}{x^\beta}$$

$$\left| x^{\alpha-1} \sin \frac{1}{x^\beta} \right| \leq |x^{\alpha-1}| \longrightarrow 0 \text{ only}$$

if $\alpha - 1 > 0$

$\alpha > 1$.

$\Rightarrow f'(0)$ exist if $\alpha > 1$.

(b) If $\alpha > \beta \Rightarrow f$ is of bounded ~~of~~ variation on $[0, 1]$.

Q. find all $\gamma \in \mathbb{R}$ such that if f is any continuous function on $[1,3]$ with

$$\int_1^3 f(x)\,dx = 1$$, then there exist some $x_0 \in (1,3)$ with $f(x_0) = \gamma$.

Soln $f(x) = 1/2$

Q. find the domain of convergence of the series

$$\sum_{n=1}^{\infty} \frac{n\,4^n}{3^n} x^n (1-x)^n .$$

Soln. $|x - x^2| < \frac{3}{4}$ by Cauchy hadamard test

By $\Rightarrow |x(1-x)| \frac{4}{3} < 1 \Rightarrow$ Given series is converges.

$\Rightarrow -\frac{1}{2} < x < 3/2$

Q. S^1 is homeomorphic to S^2. True / False

Sol False.

Suppose that $f: S^1 \longrightarrow S^2$ is a homeomorphism, and $x, y \in S^1$ with $x \neq y$

Then

$$f: S^1 \setminus \{x, y\} \longrightarrow S^2 \setminus \{f(x), f(y)\}$$

would also be a homeomorphism.

However, $S^1 \setminus \{x, y\}$ is not connected,

while $S^2 \setminus \{f(x), f(y)\}$ is connected.

Q. Is $|x|$ is a polynomial? Yes/No.

Soln No because all polynomials are differentiable but $|x|$ is not differentiable at $x = 0$.

Q.

Q. find a continuous function $f : [0, \infty) \longmapsto \mathbb{R}$ which is bounded but not attain a maximum value?

soln. $f(x) = \dfrac{x}{|x|+1}$

Here f is bounded below by 0 and above by 1.

Q. find a continuous bounded function on $\mathbb{R}$ which attain maximum but not minimum.

soln
$$f(x) = \begin{cases} e^{-x} & \text{for } x \geq 0 \\ 1 & \text{if } x < 0 \end{cases}$$

f is continuous and bounded. Its maximum value 1 is attained but ~~infm~~ minimum value 0 is not attained.

Q. let $f:[0,4] \longrightarrow [1,3]$ be a differentiable function such that $f'(x) \neq 1$ for all $x \in [0,4]$. Then the function f has more than one fixed points. True / False

Sol False. $f'(x) \neq 1 \Rightarrow f'(x) < 1$ or $f'(x) > 1$.

suppose $f'(x) < 1$, then

Take, $f(x) = 6-x$. $f(0) - 0 \geq 1 > 0$ and

$f(4) - 4 \geq -1 < 0$

so $g(x) = f(x) - x$ ~~have~~ has ~~root~~ ~~some~~ root i.e $g(x) = 0$ for some $x \in (0,4)$

by intermediate theorem.

For example take $f(x) = 6-x$

$f'(x) = -1 < 1$

$f(3) = 6-3 = 3$

$f(3) = 3$ fixed point.

Suppose f has more ~~or~~ than one fixed points.

Assume two fixed points $a < b \Rightarrow f(a) = a$ and $f(b) = b$

By the mean value theorem, there is some ~~c~~ $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Since $f(b) = b$, $f(a) = a$ (more than ~~two~~ one fixed points)

$$f'(c) = \frac{b - a}{b - a}$$

$\boxed{f'(c) = 1}$ This leads to a contradiction

because $f'(c) \neq 1$.

Also, $f'(x) > 1$ is not possible because f has unique fixed point.

Q. There exist a differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) = 1$ and $f'(x) \geq ((f(x)))^2 \ \forall \ x \in \mathbb{R}$.

True / False.

False.

Soln let $f(x) \neq 0$ for all $x > 0$.

suppose if there exists a point x_0 such that $f(x_0) = 0$, then by the mean-value theorem, there exist a point $c \in (0, x_0)$ such that

$$f'(c) = \frac{f(x_0) - f(0)}{x_0 - 0}$$

$$f(x_0) - f(0) = f'(c)\, x_0$$

$f(x_0) = 0$, $f(0) = 1$ and $f'(x) \geq ((f(x)))^2$

$$\Rightarrow \quad 0 - 1 = f'(c)\, x_0$$

$$-1 \geq (f(c))^2\, x_0 \geq 0$$

$-1 \geq (f(c))^2 x_0$. This leads to a

Contradiction

$-1 \geq (f(c))^2$ not possible

Therefore $f(x) \neq 0$ for all $x > 0$.

Now, if $f'(x) \geq (f(x))^2$, then one has

$$\frac{f'(x)}{(f(x))^2} \geq 1.$$

$$\Rightarrow \frac{d\, f(x)}{(f(x))^2} \geq 1.$$

Now integrating both sides, we have

$$\int \frac{d\, f(x)}{(f(x))^2} \geq \int dx$$

$$\Rightarrow -\frac{1}{f(x)} \geq x + c$$

$$\Rightarrow -\frac{1}{f(x)} \geq x \Rightarrow \frac{-1}{x} \geq f(x)$$

~~For~~ $x < 0$

For $x > 0$, we have $f(x) < 0$

Since $f(0) = 1 > 0$, then by the intermediate theorem, we must have

$$f(x_1) = 0 \text{ for some } x_1 > 0.$$

However, we already assumed that f would never equal to 0 i.e $f(x) \neq 0$ for all $x > 0$.

Contradiction.

Q. Let (X, d) be a metric space and $A \subset X$.

Then $(A^\circ)^\circ = A^\circ$. True | False

Soln

By definition, $(A^\circ)^\circ \subset A^\circ$. — (1)

Let $x \in A^\circ$. By definition of A°, there exist $\varepsilon > 0$ such that $B(x, \varepsilon) \subset A^\circ$.

Therefore, $x \in (A^\circ)^\circ$ since $x \in B(x, \varepsilon)$

$\Rightarrow \quad A^\circ \subset (A^\circ)^\circ$ — (2)

From (1) and (2), we have

$$\boxed{(A^\circ)^\circ = A^\circ}$$

Q Give an example of a topological space (X, T) and a subset A of X which is neither open nor closed.

Soln. let $X = \mathbb{R}$ with usual topology

let $U_n = [0, 1 + \frac{1}{n})$. Then

$$A = \bigcap_{n=1}^{\infty} U_n = [0, 1]$$

Q Under the usual topology on $\mathbb{R}^3$, ~~then~~ the map $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined by

$f(x, y, z) = (x+1, y-1, z)$ is both open and closed. True / False

Soln.

$f^{-1} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined by

$$f^{-1}(x, y, z) = (x-1, y+1, z)$$

$\Rightarrow f \circ f^{-1} = f^{-1} \circ f = id_{\mathbb{R}^3}$

$\Rightarrow f$ is a homeomorphism

both f and f^{-1} are continuous

$\Rightarrow$ f is both open and ~~c~~ closed.

Q. let $f(z) = \sum_{n=-\infty}^{\infty} 5^{-|n|} z^{2n}$, then it converges

for $\frac{1}{\sqrt{5}} < |z| < \sqrt{5}$. True / False

Soln True

Refer $\sum_{n=-\infty}^{\infty} 5^{-|n|} z^{2n}$

$$= \sum_{-\infty}^{-1} 5^{-|n|} z^{2n} + \sum_{n=0}^{\infty} 5^{-|n|} z^{2n}$$

$$= \underbrace{\sum_{n=1}^{\infty} 5^{-n} z^{-2n}} + \underbrace{\sum_{n=0}^{\infty} 5^{-|n|} z^{2n}}$$

$|z| < \sqrt{5}$

$|z\cancel{^2}| > \frac{1}{\sqrt{5}}$

$|z| > \frac{1}{\sqrt{5}}$

$$\Rightarrow \frac{1}{\sqrt{5}} < |z| < \sqrt{5}$$ **Ans**

Q. Is $\frac{n}{e^n} \to 0$ as $n \to \infty$. True / False

Soln True.

$$\boxed{e^n \geq \frac{n^2}{2}}$$

$$\frac{e^n}{n} \geq \frac{n^2}{n \cdot 2}$$

$$\frac{e^n}{n} \geq \frac{n}{2}$$

$$\frac{n}{e^n} \leq \frac{2}{n}$$

$$\Rightarrow \frac{n}{e^n} \leq \frac{2^n}{e^n} = \left(\frac{2}{e}\right)^n$$

$$\lim_{n \to \infty} \left(\frac{2}{e}\right)^n = 0 \quad \text{since } e > 2$$

Q. Let f be a real valued ~~function~~ continuous function on $[0, \infty)$

such that

$$\lim_{x\to\infty} \left(f(x) + \int_0^x f(t)\,dt \right) \text{ exists.}$$

Then $\lim_{x\to\infty} f(x) = 0$. True / False

Soln True

$$\text{Let } L = \lim_{x\to\infty} \left(f(x) + \int_0^x f(t)\,dt \right)$$

Here we can write

$$f(x) + \int_0^x f(t)\,dt = \frac{\frac{d}{dx}\left(e^x \int_0^x f(t)\,dt \right)}{\frac{d}{dx} e^x}$$

$$\text{and } \lim_{x\to\infty} \int_0^x f(t)\,dt = \lim_{x\to\infty} \frac{e^x}{e^x} \cdot \int_0^x f(t)\,dt$$

$$= \lim_{x\to\infty} \frac{e^x \cdot \int_0^x f(t)\,dt}{e^x}$$

$$= L$$

$$\lim_{x\to\infty} f(x) = \left(\lim_{x\to\infty} \left(f(x) + \int_0^x f(t)\,dt\right) - \int_0^x f(t)\,dt\right)$$

$$= L - L$$

$$\boxed{\lim_{x\to\infty} f(x) = 0}$$ Ans

Q. Let $f : (0,\infty) \longrightarrow (0,\infty)$ be a montone decreasing function defined on the positive real numbers with

$\int_0^\infty f(x)dx < \infty$. Then $\lim_{x \to \infty} x f(x) = 0$

True / False

Sol^n True.

Given f is a decreasing function.

$\Rightarrow$ For all $x > 0$, the minimum of $f(c)$ is $f(x)$ where $c \in [\frac{x}{2}, x]$

$$\frac{x}{2} f(x) = \int_{x/2}^{x} f(x)\, dt = f(x) \int_{x/2}^{x} \cancel{f(x)}\, dt$$

$$\leq \int_{x/2}^{x} f(t)\, dt$$

Now using Cauchy criterion we have

$$\lim_{x \to \infty} \int_{x/2}^{x} f(t)\, dt = 0 \text{ since } \int_0^\infty f dx < \infty$$

$$x f(x) \leq \alpha \int_{x/2}^{x} f(t)\, dt$$

$$\boxed{\lim_{x \to \infty} x f(x) = 0}$$

Q. Let f be a continuous real valued function satisfying $f(x) \geq 0$ for all x, and

$$\int_0^\infty f(x)\,dx < \infty$$

Then $\frac{1}{n} \int_0^n x f(x)\,dx \longrightarrow 0$ as $n \longrightarrow \infty$

True / False :

Sol: True

Given $\int_0^\infty f(x)\,dx < \infty$ and $f(x) \geq 0$

Let $g_n(x) = \frac{x}{n} f(x)\, \chi_{[0,n]}(x)$

where $\chi_{[0,n]}$ denote the characteristic function

i.e $\chi_{[0,n]} = \begin{cases} 1 & \text{if } x \in [0,n] \\ 0 & \text{if } x \notin [0,n] \end{cases}$

$$\frac{1}{n}\int_0^n x\, f(x)\, dx = \int_0^n \frac{1}{n}\, x\, f(x)\, dx$$

$$= \int_0^\infty \frac{x}{n} \chi_{[0,n]}(x)\, f(x)\, dx$$

$$= \int_0^\infty \frac{x}{n} f(x)\, \chi_{[0,n]}(x)\, dx$$

$$= \int_0^\infty g_n(x)\, dx$$

$$0 \le \frac{x}{n} \chi_{[0,n]}(x) \le 1, \quad \text{for all } x \geq 0$$

$$\Rightarrow |g_n(x)| \le |f(x)| = f(x) \quad \text{for all } x.$$

Now using the dominated convergence theorem,
we have

$$\lim_{n\to\infty} \frac{1}{n}\int_0^n x\, f(x)\, dx = \lim_{n\to\infty} \int_0^\infty g_n(x)\, dx$$

$$= \int_0^\infty \lim_{n\to\infty} g_n(x)\, dx$$

$$= 0$$

Q. let $f: \mathbb{R} \to \mathbb{R}$ be a differentiable function ~~whose~~ whose derivative is continuous.

find $\lim_{n \to \infty} \left((n+1) \int_0^1 x^n f(x)\,dx \right)$

Soln.

$$(n+1) \int_0^1 x^n f(x)\,dx$$

$$= (n+1) \left[\frac{x^{n+1}}{n+1} f(x) \Big|_0^1 \right.$$

$$- \int_0^1 f'(x)dx \int_0^1 x^n dx$$

$$= x^{n+1} f(x) \Big|_0^1 - \int_0^1 x^{n+1} f'(x)\,dx$$

$$= f(1) - \int_0^1 x^{n+1} f'(x)\,dx$$

f' is continuous over $[0,1]$

$\Rightarrow f'$ is bounded on the interval $[0,1]$

suppose $f'(x)$ is bounded by $M \geq 0$, then

$$\left|\int_0^1 x^{n+1} f'(x)\,dx\right|$$

$$\leq \int_0^1 x^{n+1} |f'(x)|\,dx$$

$$\leq M \int_0^1 x^{n+1}\,dx$$

$$= \frac{M}{n+2} \to 0 \quad \text{as } n \longrightarrow \infty$$

$$\Rightarrow \boxed{\lim_{n\to\infty} (n+1) \int_0^1 x^n f(x)\,dx = \cancel{f(1)}\ f(1)}$$

Note :- For $0 < a < 1$, $\lim_{n\to\infty} \int_0^a (n+1)\, x^n f(x)\,dx = 0$

Q. let $K \geq 0$ be an integer and define a sequence of maps

$$f_n : \mathbb{R} \longrightarrow \mathbb{R}, \quad f_n(x) = \frac{x^k}{x^2+n}, \quad n = 1, 2, \ldots$$

For which values of K does the sequence

(i) Converge uniformly on $\mathbb{R}$?

(ii) Converge uniformly on every bounded subset of $\mathbb{R}$?

Soln. (i) Convergence is on $\mathbb{R}$ when $K = 0$ and 1 only

The sequence $\{f_n(x)\}$ converges pointwise to 0 for any value of K.

Now, if $f_n \longrightarrow 0$ uniformly in the real line, then $f_n(n)$ has to converge to 0.

$$\Rightarrow f_n(n) = \frac{n^k}{n^2+n} \longrightarrow 0 \text{ for } K < 2$$

If $K = 0$, then $f_n(x) = \frac{1}{x^2+n}$

$$f_n(n) = \frac{1}{n^2+n} \leq 1/n$$

$$\Rightarrow f_n(x) \leq \frac{1}{n}$$

$$\sup_{x \in \mathbb{R}} |f_n(x) - f(x)|$$

$$= \sup_{x \in \mathbb{R}} |f_n(x) - 0|$$

$$\lim_{n \to \infty} \sup_{x \in \mathbb{R}} |f_n(x)| = 0$$

Therefore, f_n converges uniformly on $\mathbb{R}$

If $K = 1$, $f_n(x) = \frac{x^1}{x^2+n} = \frac{x}{x^2+n}$

$$f_n'(x) = \frac{n-x^2}{(x^2+n)^2}$$

So $|f_n|$ attains its maximum at $x = \sqrt{n}$,

where $\sup|f_n| = f_n(\sqrt{n}) = \frac{\sqrt{n}}{n+n}$

$$\sup_{x \in \mathbb{R}} |f_n - f| = \frac{1}{2\sqrt{n}}$$

$$\lim_{n \to \infty} \sup |f_n| \longrightarrow 0 \quad \text{as } n \to \infty$$

Therefore, f_n converges uniformly on $\mathbb{R}$.

If $K = 2$, $f_n(x) = \dfrac{x^2}{x^2 + n}$

$$f(x) = \lim_{n \to \infty} f_n(x) = 0$$

$$f_n(n) = \frac{n^2}{n^2 + n} \leq \frac{n^2}{n^2} = 1$$

$$\sup |f_n(x) - f(x)| = \sup |f_n(x)|$$
$$= 1$$

Hence f_n doesn't converge uniformly on $\mathbb{R}$

for $K \geq 3$, $\sup |f_n| = \infty$ as $n \longrightarrow \infty$

$\Rightarrow$ Convergence is not uniform on $\mathbb{R}$.

(ii) For <u>bounded subset of $\mathbb{R}$</u>

Let $|x| \leq M$

$$\Rightarrow f_n(x) = \frac{x^k}{x^2+n} \leq \frac{M^k}{n}$$

$\Rightarrow$ For all value of k, Convergence is uniform on bounded sets.

Q. Let $f_1, f_2 \dots$ be continuous function on $[0,1]$ satisfying $f_1 \geq f_2 \geq \dots$ and such that $\lim_{n \to \infty} f_n(x) = 0$ for each x.

Then sequence $\{f_n\}$ converge to 0 uniformly on $[0,1]$. True / False

Sol. True by using Dini's theorem

Q. let $\{a_n\}$ be a sequence of positive numbers such that $\lim_{n\to\infty} a_n = 0$, ~~but the~~ then series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges

True / False

Sol^n False, Take $a_n = \begin{cases} \frac{2}{n+1} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$

$\lim_{n\to\infty} a_n = 0$

The $(2n-1)^{th}$ partial sum of $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ is equal to the n^{th} partial sum of $\sum_{n=1}^{\infty} \frac{1}{n}$.

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges $\Rightarrow \sum_{n=1}^{\infty} (-1)^{n+1} a_n$ also diverges

For $a > 0$, we have

$$\boxed{\lim_{x \to 0} \frac{a^x - 1}{x} = \log a}$$

proof: $\lim_{x \to 0} \frac{a^x - 1}{x} = \cancel{\log a} \cancel{\lim_{t \to 0}}$

$$= \lim_{x \to 0} \frac{a^x \log a}{1}$$

$$= a^0 \log a$$

$$= \log a$$

Q. find all pairs of integers a and b satisfying $0 < a < b$ and $a^b = b^a$

Soln $a = 2, b = 4$

Q. Let $A_1 \geq A_2 \geq \cdots \geq A_k \geq 0$.

Evaluate

$$\lim_{n\to\infty} \left(A_1^n + A_2^n + \dots + A_k^n\right)^{1/n}$$

Sol. $A_1^n \leq A_1^n + \dots + A_k^n \leq kA_1^n$, so

we have

$$A_1 = \lim_{n\to\infty} (A_1^n)^{1/n} \leq \lim_{n\to\infty} (A_1^n + \dots + A_k^n)^{1/n}$$

$$\leq \lim_{n\to\infty} (k A_1^n)^{1/n}$$

$$= A_1$$

$$\Rightarrow \lim_{n\to\infty} (A_1^n + A_2^n + \dots + A_k^n)^{1/n} = A_1.$$

Q. find for which values of $\alpha \in \mathbb{R}$ such that $f_n(x)$ converge uniformly in $[0,\infty)$ where $f_n(x) = n^\alpha x e^{-nx}$

Sol

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} n^\alpha \frac{x}{e^{nx}}$$

$$= 0 = f(x)$$

$\therefore f_n(x)$ is pointwisely convergent to the zero function on the interval $[0,\infty)$

$$f_n'(x) = n^\alpha e^{-nx}(1-nx)$$

$$= 0$$

$\Rightarrow x = 1/n$

$\Rightarrow f_n$ attain maximum at $x = 1/n$

$$\sup_{x=1/n} |f_n(x) - f(x)| = \|f_n\|_\infty$$

$$= \cancel{n^\alpha e^{-n\cdot 1/n}(\tfrac{1}{n})}$$

$$= n^{\alpha} e^{-n x_n} \cdot \frac{1}{n}$$

~~x_0~~ .

$$\| f_n \|_\infty = f_n(x_n)$$

$$= n^{\alpha-1} e^{-1} \longrightarrow 0 \text{ as } n \to \infty$$

when. $\alpha - 1 < 0 \Rightarrow \alpha < 1$.

Q. Is $f_n(x) = \frac{x}{1+nx^2}$ Converges uniformly for $x \in \mathbb{R}$? True / False

Soln. True

$$\lim_{n \to \infty} f_n(x) = f(x) = 0$$

$$f_n'(x) = \frac{1\cdot(1+nx^2) - 2nx^2}{(1+nx^2)^2}$$

$$= \frac{1-nx^2}{(1+nx^2)^2} = 0$$

$$\Rightarrow \quad x = \frac{1}{\sqrt{n}}$$

i.e $f_n(x) = \frac{x}{1+nx^2}$ attains the maximum value at $x = \frac{1}{\sqrt{n}}$

$$\sup_{x \in \mathbb{R}} |f_n(x) - 0| = \sup_{x \in \mathbb{R}} \left| \frac{x}{1+nx^2} \right|$$

$$= \left(\left| \frac{x}{1+nx^2} \right| \right)_{x = \frac{1}{\sqrt{n}}}$$

$$= \frac{1}{2\sqrt{n}} \to 0 \text{ as } n \to \infty$$

$\Rightarrow$ $f_n(x)$ converge uniformly to $f(x)$.

Q. let $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a real valued function.
If the derivative f' is bounded on $\mathbb{R}$, then
f is uniformly continuous on $\mathbb{R}$. True / False

Soln True

Since f' is bounded then there's $M > 0$
such that $|f'(x)| \leq M \ \forall x \in \mathbb{R}$

By using mean value theorem, we have

$$\frac{f(x) - f(y)}{|x-y|} \leq M$$

$$\Rightarrow f(x) - f(y) \leq |x-y| \cdot M \quad \forall x, y \in \mathbb{R}$$

This is the definition of lipschitz continuity.
In other word, if $f'(x)$ is bounded, then $f(x)$
is a lipschitzian function.
Since lipschitzian functions are uniformly ~~continuous~~ continuous,
$\Rightarrow$ $f(x)$ is uniformly continuous on $\mathbb{R}$.

Converse Statement

On interval $[0,\infty)$, these statements may not hold.

$\Rightarrow$ $f(x) = \sqrt{x}$, $f'(x)$ is not bounded.

$f(x) = x^{1/3}$ is uniformly continuous on $\mathbb{R}$ but $f'(x)$ is not bounded.

Q. For which real number x does the infinite series

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{n^x}$$

Soln.:

$$\frac{\sqrt{n+1} - \sqrt{n}}{n^x} \sim \frac{1}{n^{x+1/2}}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^{x+1/2}}$$ is converges when $x > 1/2$

Q. let $f(x) = x \log\left(1 + \frac{1}{x}\right)$, for $0 < x < \infty$

① Is f ~~is~~ strictly monotonically increasing?

② find $\lim f(x)$ as $x \to 0$ and $x \to \infty$

Soln

~~Gi~~ Given $f(x) = x \log(1 + 1/x)$

① Yes, $f(x) = \log(1 + 1/x)^x$

$$\Rightarrow \left(1 + \frac{1}{x}\right)^x = e^{f(x)}$$

$\left(1 + \frac{1}{x}\right)^x$ is an increasing function

$\Rightarrow e^{f(x)}$ is an increasing function

$\Rightarrow$ f is an increasing function.

(ii) $$\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$$

$$\Rightarrow \lim_{x \to \infty} f(x) = 1$$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\log(x+1) - \log x}{1/x}$$

$$= \lim_{x \to 0} \frac{\frac{1}{(x+1)} - 1/x}{-1/x^2}$$

$$= \lim_{x \to 0} \frac{x - x - 1}{x(x+1)} \times x^2$$

$$= 0$$

Q. Let $M_{n\times n}$ denote the vector space of $n\times n$ real matrices for $n \geq 2$.

Let $\det : M_{n\times n} \longrightarrow \mathbb{R}$ be the determinant map

Then the derivative of det at $A \in M_{n\times n}$ is zero if and only if A has rank $\leq n-2$. True | False

Sol True

For $1 \leq i, j \leq n$, we have

$$\det X = \sum_{k=1}^{n} (-1)^{k+j} x_{kj} \det X_{kj}$$

where X_{kj} denotes the k, j - Cofactor of X.

So $$\frac{d\det(X)}{dx_{ij}} = (-1)^{i+j} \det X_{ij}$$

$$= 0 . \det X_{11} + \dots + (-1)^{i+j} \det X_{ij}$$

$$\dots + 0 . \det X_{k,j}$$

$$\Rightarrow \frac{\partial \det(X)}{d x_{ij}} = (-1)^{i+j} \det X_{ij}$$

Thus, X is a critical point of det if and only if $X_{ij} = 0$ for every i and j.

X_{ij} denotes the i-j cofactor of X

so minor element of a_{ij} is

$$M_{ij} = \Big(\quad \Big)_{(n-1)\times(n-1)}$$

$\Rightarrow$ $\det X_{ij} \neq 0$ if

$\det M_{ij} = 0$ if Rank $M_{ij} < n-1$

$\Rightarrow \det X_{ij} = 0$ if Rank $X \leq n-2$ or Rank $X < n-1$

$\Rightarrow$ $X = A$ has rank $\leq n-2$

Q. Define $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by $f(x,0) = 0$

and $f(x,y) = \left(1 - \cos \frac{x^2}{y}\right) \sqrt{x^2+y^2}$

for $y \neq 0$

Q. (1) Is f continuous at $(0,0)$

Sol. Yes.

$$\left|1 - \cos \frac{x^2}{y}\right| \leq 1 + \left|\cos \frac{x^2}{y}\right|$$

$$\leq 1 + 1$$

$$= 2 \quad \text{for all } x, y \in \mathbb{R} \text{ and } y \neq 0$$

$$\Rightarrow |f(x,y)| \leq 2\sqrt{x^2+y^2}$$

$$\Rightarrow \lim_{(x,y) \to (0,0)} f(x,y) = 0$$

Q. Calculate all the directional derivatives of f at $(0,0)$.

Sol.

$$D_{(x,y)} f(0,0) = \lim_{h \to 0} \frac{f(hx, hy) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{f(hx, hy)}{h}.$$

$$= \lim_{h \to 0} \frac{1}{h} \frac{\left[1 - \cos\left(\frac{h^2x^2}{hy}\right)\right] \sqrt{h^2x^2 + h^2y^2}}{h}.$$

$$= \lim_{h \to 0} \left(1 - \cos\left(h\frac{x^2}{y}\right)\right\} \sqrt{x^2+y^2}$$

$$D_{(x,y)} f(0,0) = \begin{cases} 0 & \text{if } y \neq 0 \\ \text{undefined} & \text{if } y = 0 \end{cases}$$

$\Rightarrow$ Directional derivative at $(0,0)$ is 0 that is any direction of unit vector is not parallel to x-axis.

$\Rightarrow$ $D_{(x,y)} f(0,0)$ is undefined in the direction of the unit vector $(1,0)$ since $(1,0)$ is parallel to x-axis.