Q

Theorem :-

For any function $f(z)$ and any curve γ, we have

$$\left| \int_{\gamma} f(z)\,dz \right| \leq \int_{\gamma} |f(z)|\,|dz|$$

Here $dz = \gamma'(t)\,dt$ and $|dz| = |\gamma'(t)|\,dt$

Sol:

$$\left| \int_{\gamma} f(z)\,dz \right| = \left| \int_a^b f(\gamma(t))\,\gamma'(t)\,dt \right|$$

$$\leq \int_a^b |f(\gamma(t))|\,|\gamma'(t)|\,dt$$

$$= \int_{\gamma} |f(z)|\,|dz|$$

Theorem: If $|f(z)| < M$ on C then

$$\boxed{\left| \int_C f(z)\,dz \right| \leq M \cdot (\text{length of } C)}$$

proof: let $\gamma(t)$, with $a \leq t \leq b$ be a parametrization of C. using the triangle inequality

$$\left| \int_C f(z)\,dz \right| \leq \int_C |f(z)|\,|dz|$$

$$= \int_a^b |f(\gamma(t))|\,|\gamma'(t)|\,dt$$

$$\leq \int_a^b M\,|\gamma'(t)|\,dt$$

$$= M \cdot (\text{length of } c)$$

Here $\gamma'(t)\,dt = \sqrt{(x')^2 + (y')^2}\,dt = ds.$

Q. Does there exist a function f, analytic in the punctured plane $\mathbb{C} \setminus \{0\}$, such that

$$|f(z)| \geq \frac{1}{\sqrt{|z|}} \text{ for all non-zero } z \text{ ?}$$

Soln.

Suppose such a function f exist

then $g = 1/f$ is also analytic on $\mathbb{C} - \{0\}$, and satisfies $|g(z)| \leq \sqrt{|z|}$.

So g is bounded near 0 i.e

g is bounded on $\{z \mid 0 < |z| < 1\}$.

$\Rightarrow g$ has a removable singularity at 0, and extends as an analytic function over the Complex plane.

Fix z, choose $R > |z|$, and let C_R be the circle with centre 0 and radius R, where R is arbitrary.

then $g'(z) = \frac{1}{2\pi i} \int_{C_R} \frac{g(\omega)}{(\omega - z)^2} d\omega$

$$|g'(z)| \leq \frac{1}{|2\pi|} \int_{C_R} \frac{|g(\omega)|\,|d\omega|}{(w - |z|)^2}$$

$$\leq \frac{1}{2\pi} \cdot \frac{\sqrt{R} \cdot 2\pi R}{(R - |z|)^2}$$

Using the theorem : If $|f(z)| < M$ on C_R then

$$\left| \int_{C_R} f(z)\,dz \right| \leq M \cdot (\text{length of } C_R)$$

letting ~~Now~~, $R \longrightarrow \infty$ $\Rightarrow$ $g'(z) = 0$ everywhere

ie $g' \equiv 0$

Since $g(0) = 0$, we get that

$g(z) = 0$ for all $z \in \mathbb{C}$.

$\Rightarrow g$ is constant $\Rightarrow f$ is constant.

But this contradicts the hypothesis

$|f(z)| \geq \frac{1}{\sqrt{|z|}}$ for small z, so no such function exist.

Q. If Ω is connected. $f : \Omega \longrightarrow \mathbb{C}$ is holomorphic, and e^f is constant, then f is itself constant. True/False

Sol

e^f is constant $\Rightarrow$ $(e^f)' = 0$

$\Rightarrow$ $f'(z)\, e^f = 0$

$\Rightarrow$ $f'(z)\, e^{f(z)} = 0$

Since e^f is never zero, $f'(z) = 0$ on Ω

and hence f is a constant.

Residue at infinity :

Let $f(z)$ be a analytic function in D ~~excep~~ except $a_1, a_2, \ldots, a_n$ then

$$\boxed{\operatorname*{Res}_{z=\infty} f(z) + \operatorname*{Res}_{z=a_1} f(z) + \ldots + \operatorname*{Res}_{z=a_n} f(z) = 0}$$

$$\operatorname*{Res}_{z=\infty} f(z) = -\left[\operatorname*{Res}_{z=a_1} f(z) + \ldots + \operatorname*{Res}_{z=a_n} f(z)\right]$$

$$= -\left[\sum_{i=1}^{n} \operatorname*{Res}_{z=a_i} f(z)\right]$$

Q. Evaluate the residue of $f(z) = \dfrac{z^2}{(z-1)(z-2)}$ at $z=\infty$

Soln

$$f(z) = \frac{z^2}{(z-1)(z-2)}$$

$$\operatorname*{Res}_{z=1} f(z) = \lim_{z \to 1} (z-1) \frac{z^2}{(z-1)(z-2)}$$

$$= -1$$

$$\operatorname*{Res}_{z=2} f(z) = \lim_{z \to 2} \frac{(z-2)z^2}{(z-1)(z-2)} = 4$$

$$\operatorname*{Res}_{z=+\infty} f(z) = -(-1+4)$$
$$= -3$$

⇒ Suppose the function $f(z)$ has an isolated singularity at $z = z_0$ $(\neq \infty)$. Then in some deleted neighbourhood of z_0 $\{0 < |z - z_0| < \delta\}$ $f(z)$ can be represented by laurrent series of the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots$$

Then, the coefficient of $\frac{1}{(z-z_0)}$ is called the residue of $f(z)$ at z_0 and it is denoted by $\text{Res}(f; z_0) = b_1$

$$\Rightarrow \boxed{\begin{aligned} \int_C f(z)\,dz &= 2\pi i \,\text{Res}(f; z_0) \\ &= 2\pi i b_1 \end{aligned}}$$

where C is any positively oriented simple closed rectifiable enclosing z_0 and contained in the neighbourhood.

Suppose that $z = \infty$ is an isolated singularity of $f(z)$. Then the residue of $f(z)$ at $z = \infty$ is defined as follows

$$\boxed{\mathrm{Res}(f;\infty) = -\frac{1}{2\pi i}\int_C f(z)\,dz}$$

where C is any positively oriented simple closed contour outside of which the function f is analytic and does not have any singularity other than the point at infinity.

Cauchy's Residue Theorem :

Suppose that $f(z)$ is analytic inside and on a simple closed contour C except for isolated singularities at $z_1, z_2, \dots, z_n$ inside C. Then

$$\boxed{\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f; z_k)}$$

Q. Evaluate

$$\int_{|z|=2} f(z)\,dz \quad \text{where } f(z) = \frac{e^z}{z(z-1)^2}$$

Soln:

$$\int_{|z|=2} f(z)\,dz = \int_{|z|=2} \frac{e^z}{z(z-1)^2}\,dz$$

$$= 2\pi i \,[\,\text{Res}(f,0) + \text{Res}(f:1)\,]$$

$$\text{Res}(f,0) = \lim_{z \to 0} z f(z) = \lim_{z \to 0} \frac{e^z}{(z-1)^2}$$

$$= 1$$

$$\text{Res}(f,1) = \lim_{z \to 1} \frac{d}{dz} (z-1)^2 f(z)$$

$$= \lim_{z \to 1} \frac{d}{dz}\left(\frac{e^z}{z}\right)$$

$$= \lim_{z \to 1} \frac{z e^z - e^z}{z^2} = 0$$

$$\int_{|z|=2} f(z)\,dz = 2\pi i(1+0)$$

$$= 2\pi i$$

Q. Determines the residues of each of their isolated singularities in the extended complex plane.

(1) $\frac{1}{z^3 - z^5} = f(z)$

Note : $(z-1) = -(1-z)$

Sol^n $f(z) = \frac{1}{z^3(1-z^2)} = \frac{1}{z^3(1-z)(1+z)}$

$\underset{z=1}{\cancel{\mathrm{Res}}}$ $\cancel{\frac{1}{z}}$ $\mathrm{Res}(f,1) = \lim_{z \to 1} (z-1) f(z) =$

$= -\frac{1}{2}$

$\mathrm{Res}(f,-1) = \lim_{z \to -1} (z+1) \frac{1}{z^3 - z^5}$

$= -1/2$

$\mathrm{Res}(f,0)$ = Coefficient of $(1/z)$ in the laurent's expansion of the function around $z = 0$

= coefficient of $(1/z)$ in $\frac{1}{z^3}(1 + z^2 + z^4 + \dots)$

$= 1$

$$\text{Res}(f, \infty) = -\left(1 - \tfrac{1}{2} - \tfrac{1}{2}\right)$$

$$= 0$$

Q <u>Evaluate</u>

$$\oint_C \frac{\sin z}{z^4}\,dz \quad \text{where} \quad C : |z| = 1.$$

Soln.

$$f(z) = \frac{\sin z}{z^4} = \frac{1}{z^3} - \frac{1}{3!\,z} + \frac{z}{5!} - \frac{z^3}{7!} + \cdots$$

$$\text{Res}(f, 0) = -\tfrac{1}{6}$$

$$\oint_C \frac{\sin z}{z^4}\,dz = 2\pi i \,\text{Res}(f, 0)$$

$$= 2\pi i \left(-\tfrac{1}{6}\right)$$

$$= -\frac{\pi i}{3}$$

Essential singularity :-

A function $f(z)$ can be expanded as a laurent series about a singularity z_0 as

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

The second sum (the principal part) has infinitely many terms, then the singularity $z = z_0$ of $f(z)$ is called an essential singularity

Example :- $e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{z^n n!}$ has an essential singularity in 0.

Q. let $f : \mathbb{C} \setminus \{0\} \longrightarrow \mathbb{C}$ be an analytic function such that $\int_\gamma f(z)\,dz = 0$ for every simple closed curved γ on $\mathbb{C} \setminus \{0\}$.

Then f cannot have essential singularity at 0. True / False

Soln. False, Take $f(z) = \frac{1}{z^2} e^{1/z}$

f have essential singularity at 0 because we will get infinitely many non-zero terms in negative power of z ~~for~~ for $f(z)$.

$$f(z) = \frac{1}{z^2} e^{1/z}$$

$$= \frac{1}{z^2}\left(1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \cdots\right)$$

$$\mathrm{Res}(f, 0) = 0$$

$$\int_\gamma \frac{1}{z^2} e^{1/z}\,dz = 0$$

Q. let $f: \mathbb{C} \longrightarrow \mathbb{C}$ be an entire function and let $a > 0$ and $b > 0$ be constants.

If $|f(z)| \leq a\sqrt{|z|} + b$ for all z, then f is constant. True / False

Soln: True.

$$f(z) = \sum_{n=0}^{\infty} c_n z^n \quad \text{for all } z \in \mathbb{C}.$$

let C_R be the circle with ~~c~~ centre 0 and radius $R > 0$. Then we have ~~fat~~ integral representation ~~fo~~ for coefficients

$$c_n = \frac{1}{2\pi i} \int_{C_R} \frac{f(z)}{z^{n+1}} \, dz$$

$$|c_n| \leq \frac{1}{2\pi} \int_{C_R} \frac{|f(z)| \, |dz|}{|z|^{n+1}}$$

$$\leq \frac{1}{2\pi} \frac{2\pi R \, (b + aR^{1/2})}{R^{n+1}}$$

$$= \frac{1 \, (aR^{1/2} + b)}{R^n}$$

$$|C_n| \leq \frac{aR^{1/2} + b}{R^n}$$

Hence for $n \in \mathbb{N}$ we obtain

$$|C_n| \leq \lim_{R \to \infty} \frac{aR^{1/2} + b}{R^n}$$

$$= 0$$

which implies $c_n = 0$ for all $n \in \mathbb{N}$.

Therefore $$f(z) = c_0 + \sum_{n=1}^{\infty} c_n z^n$$

$$= c_0 + 0$$

$$f(z) = c_0 = \text{Constant}$$

Q. Let $f: \mathbb{C} \longrightarrow \mathbb{C}$ be an entire function and let $a > 0$ and $b > 0$ be constant. If $|f(z)| \leq a|z|^{5/2} + b$ for all z. Then f is constant. True / False

Soln: False. Let C_R be circle with centre 0.

Take $f(z) = \sum_{n=0}^{\infty} c_n z^n$

$$c_n = \frac{1}{2\pi i} \int_{C_R} \frac{f(z)}{z^{n+1}} dz.$$

$$|c_n| \leq \frac{1}{2\pi} \int_{C_R} \frac{|f(z)|\,|dz|}{|z|^{n+1}}$$

$$\leq \frac{1}{2\pi} 2\pi R \frac{(aR^{5/2} + b)}{R^{n+1}}$$

$$|c_n| \leq \frac{aR^{5/2} + b}{R^n}$$

For $n \in \mathbb{N}$, we have

$$|c_n| \leq \lim_{R \to \infty} \frac{aR^{5/2} + b}{R^n} = 0$$

$\Rightarrow \quad c_n = 0$ for all $n \geq 3$

$$\lim_{R \to \infty} \frac{aR^{5/2} + b}{R^2} \neq 0 \quad \text{or}$$

$$\lim_{R \to \infty} \frac{aR^{5/2} + b}{R} \neq 0$$

It follows that $f(z)$ is a polynomial of atmost degree 2

i.e $\quad f(z) = c_0 + c_1 z + c_2 z^2 + 0 \cdot z^3 + 0 \cdot z^4 + \ldots$

Q: Let the sequence $a_0, a_1, \ldots$ be ~~defined~~

defined by the equation

$$1 - x^2 + x^4 - x^6 + \cdots = \sum_{n=0}^{\infty} a_n (x-3)^n$$

find $\lim_{n\to\infty} \sup(|a_n|^{1/n})$

Soln. We know that radius of convergence, R is define by

$$\frac{1}{R} = \lim_{n\to\infty} \sup |a_n|^{1/n}$$

$$R = \frac{1}{\lim_{n\to\infty} \sup |a_n|^{1/n}}$$

$$1 - x^2 + x^4 - x^6 + \cdots = \frac{1}{1+x^2}$$

which has singularities at $\pm i$, the radius of convergence of power series $\sum_{n=0}^{\infty} a_n (x-3)^n$ is the distance from 3 to $\pm i$, $|3 \mp i| = \sqrt{10}$

$\Rightarrow \quad R = \sqrt{10}$

Therefore, $\limsup\limits_{n\to\infty} \left(|a_n|^{1/n}\right) = \dfrac{1}{R}$

$= \dfrac{1}{\sqrt{10}}$

Also, $\dfrac{1}{1+x^2}$ has poles at $x = \pm i$

Therefore, the radius of convergence is the distance from 3 to the closest of these poles.

Q: Let f be an analytic function such that

$$f(z) = 1 + 2z + 3z^2 + \cdots \quad \text{for } |z| < 1$$

Define a sequence of real number $a_0, a_1, a_2, \ldots$

by $f(z) = \sum_{n=0}^{\infty} a_n (z+2)^n$

What is the radius of convergence of the series

$$\sum_{n=0}^{\infty} a_n z^n \,?$$

Sol^n For $|z| < 1$, we have

$$\sum_{n=1}^{\infty} n z^{n-1} = \left(\sum_{n=0}^{\infty} z^n \right)'$$

$$= \frac{d}{dz} \left(\sum_{n=0}^{\infty} z^n \right) = \frac{d}{dz} \left(\frac{1}{1-z} \right)$$

$$1 + 2z + 3z^2 + \cdots = \sum_{n=0}^{\infty} n \cdot z^{n-1}$$

$$= \frac{d}{dz} \left(\frac{1}{(1-z)} \right) = \frac{1}{(1-z)^2}$$

$$\boxed{\frac{d}{dx} x^n = n x^{n-1}}$$

$f(z)$ has poles at $z = 1$ of order 2

$\Rightarrow$ Radius of convergence is the distance from -2 to the closest of these pole

$$|-2-1| = |-3| = 3$$

~~Rec~~ or

$$|-2-1+0i| = \sqrt{3^2+0^2}$$
$$= \sqrt{3^2}$$
$$= 3$$

Radius of convergence $R = 3$

power series expansion of f centred at -2 will have a radius of convergence equal to the distance between -2 and 1.

Hence $R = 3$.

Q. Find the number of roots of

$z^7 - 4z^3 + 11 = 0$ which lie between

the two circles $|z| = 1$ and $|z| = 2$.

Solⁿ:

on the circle $|z| = 1$, we have

$$f(z) = 11$$
$$g(z) = z^7 - 4z^3$$

$$|g(z)| < |f(z)|$$

By Rouche theorem $f(z)$ and $f(z) + g(z)$ have same zeros inside $|z| = 1$

But $f(z)$ has no zeros in the unit disc.

For $|z| = 2$

$$f(z) = z^7, \quad g(z) = |-4z^3 + 11|$$
$$\leq 43$$

$$f(z) = 128 = 2^7 = 2^5 \cdot 2^2$$
$$= 32 \times 4$$

By Rouche theorem, $f(z)$ and $f+g$ have seven zeros inside the disc $\{z \mid |z| < 2\}$

Therefore $p(z) = z^7 - 4z^3 + 11$ has 7 ~~zeroes~~ zeroes.

Q. Do there exist functions $f(z)$ and $g(z)$ that are analytic at $z=0$ and that satisfy

(1) $f(1/n) = f(-1/n) = 1/n^2$, $n = 1, 2, \ldots$

(2) $g(1/n) = g(-1/n) = 1/n^3$, $n = 1, 2, 3, \ldots$

Sol. (1) $f(z) = z^2$ is entire and satisfies

$$f(1/n) = f(-1/n) = 1/n^2$$

(2) No, no such entire function exist. by identity theorem.

Q. Is ~~ft~~ $f : \mathbb{C} \longrightarrow \mathbb{C}$ defined by

$$f(z) = z^3 \text{ injective ?}$$

Soln. No, Take $z_1 = e^{2\pi i/3}$, $z_2 = e^{-2\pi i/3}$

$$f(z_1) = f(e^{2\pi i/3}) = 1$$

$$f(z_2) = f(e^{-2\pi i/3}) = 1$$

$$f_1 = f_2 \quad \text{but } z_1 \neq z_2$$

$\Rightarrow$ f is not injective

Q. Determine the group $\mathrm{Aut}(\mathbb{C})$ of all one-one analytic maps of $\mathbb{C}$ onto $\mathbb{C}$?

Soln. $\mathrm{Aut}(\mathbb{C}) = \{\text{analytic bijection } f : \mathbb{C} \rightarrow \mathbb{C}\}$

$$\mathrm{Aut}(\mathbb{C}) = \{f \mid f(z) = az + b,\ a \neq 0\}$$

Q. The function $f(z) = \frac{1}{z^2}$ goes to 0 as $z \longrightarrow \infty$ but it is not constant. Does this contradict Liouville's theorem?

Soln. No, Liouville's theorem requires the function be entire. $f(z)$ has a singularity at the origin, so it is not requi entire.

Q. Suppose $f(z)$ is analytic on and inside a simple closed curve γ. If f has n zeroes inside γ then $f'(z)$ has $n-1$ zero inside γ. True / False

Soln. False, Take $f(z) = e^z - 1$

$f(z)$ has 3 zeros inside the Circle $|z| = 3\pi$ $(0, \pm 2\pi)$

i,e $e^z = 1 \Rightarrow z = k\pi i$

$k = 0, 2\pi, -2\pi$

But $f'(z) = e^z$ has no zeroes.

Q. Suppose $f(z)$ is analytic on and inside the unit circle. Suppose also that $|f(z)| < 1$ for $|z| = 1$. Then $f(z)$ has exactly one fixed point $f(z_0) = z_0$ inside the unit disc. True / False

Sol^n ~~Let~~ True

let $h(z) = f(z) - z$

$g(z) = z$

$|h(z)| < |g(z)|$

used the formula

$$\boxed{\big||z_1| - |z_2|\big| \le |z_1 - z_2|}$$

$\Rightarrow \big||f(z)| - |z|\big| < |g(z)|$

$\Rightarrow |\delta - 1| < 1$ where $\delta = |f(z)| < 1$

By Rouche theorem, $g(z)$ and $g(z) + h(z)$ have same zeroes.

$\Rightarrow g(z)$ has 1 zero in the unit disc.

$$g(z) + h(z) = z + f(z) - z$$

$$= f(z)$$

$\Rightarrow$ $f(z)$ has exactly one ~~zeroes~~ zero.

i.e $f(z)$ has exactly one zero in the unit disk.

Q. Which of the following are meromorphic in the whole plane.

(1) z^5

(2) $z^{5/2}$

(3) $e^{1/z}$

(4) $\frac{1}{\sin z}$

Sol^n (1) z^5 is entire $\Rightarrow$ meromorphic in the whole plane.

$f(z) = z^5$ is meromorphic

(3) $f(z) = e^{1/z}$ is not a meromorphic because the singularity at $z=0$ is an essential singularity, not a finite pole.

(4) $f(z) = \frac{1}{\sin z}$ is meromorphic

$\sin z$ has simple zeros at $n\pi$ for all integer n. So $\frac{1}{\sin z}$ has simples poles at these points.

A meromorphic function on an open subset D of the complex plane is a function that is holomorphic on all of D except for a set of isolated points, which are poles of the function.

HQ Is it true that $|a^b| = |a|^{|b|}$?

Soln. False Take $|e^i| = 1$ but $|e|^{|i|} = e^1 = e$

$$|e^i| \neq |e|^{|i|}$$

Q Solve $z^4 - i = 0$

Soln

$$i = e^{i(2n\pi + \pi/2)} \times \frac{1}{4}$$

$$z = i^{1/4}$$

$$= e^{i(\pi/8 + n\pi/2)}$$

$$\boxed{z = \pm e^{i\pi/8}, \ \pm i e^{i\pi/8}}$$

Q Evaluate

$$\int_{|z|=2} \tan z \, dz.$$

Sol^n

$$\int_{|z|=2} \tan z \, dz = 2\pi i \left[\underset{z=\pi/2}{\text{Res}} \tan z + \underset{z=-\pi/2}{\text{Res}} \tan z \right]$$

z_0

$$\underset{z=\pi/2}{\text{Res}} \tan z = \lim_{z\to\pi/2} (z-\pi/2) \tan z$$

$$= \lim_{z\to\pi/2} (z-\pi/2) \frac{\sin z}{\cos z}$$

$$= \lim_{z\to\pi/2} \sin z \lim_{z\to\pi/2} \left(\frac{z-\pi/2}{\cos z} \right)$$

$$= \lim_{z\to\pi/2}$$

$$= 1 \lim_{z\to\pi/2} \frac{\frac{d}{dz}(z-\pi/2)}{\frac{d}{dz}(\cos z)}$$

$$= 1 \cdot \lim_{z \to \pi/2} \frac{1}{-\sin z}$$

$$= 1 \cdot (-1) \quad = -1$$

Similarly, $\underset{z = -\pi/2}{\text{Res}} \tan z = -1 \quad$ since $(-1) \cdot \times \dfrac{1}{(-1)\sin(-\pi/2)}$

$$= -1$$

$$\int_{|z|=2} \tan z \, dz = 2\pi i \left[-1 + (-1)\right]$$

$$= -4\pi i$$

Q: How many zeroes the complex polynomial

$3z^9 + 8z^6 + z^5 + 2z^3 + 1$ have in the

annulus $1 < |z| < 2$?

Soln:

For $|z| = 1$

$f(z) = 8z^6$

$g(z) = 3z^9 + z^5 + 2z^3 + 1$

& $|g(z)| < |f(z)|$

By Rouche theorem, f and $f+g$ have

6 zeroes inside disc $\{z / |z| \leq 1\}$.

For $|z| = 2$

$f(z) = 3z^9$

$g(z) = 8z^6 + z^5 + 2z^3 + 1$

$|g(z)| < |f(z)|$

By Rouche theorem. f has 9 roots

with $|z| < 2$.

Therefore the number of roots in

the annulus $1 < |z| < 2$ $= 9 - 6$

$= 3$

Q Suppose $p \in P(\mathbb{F})$ has degree n.

Then p has n distinct roots if and only if p and its derivative p' have no roots in common. True / False

Soln True

let $p(x) = (x-\lambda)^k q(x)$ where $q(\lambda) \neq 0$, then

$$p'(x) = k(x-\lambda)^{k-1} q(x) + (x-\lambda)^k q'(x)$$

Now $(x-\lambda) \mid p(x)$ and $(x-\lambda) \mid p'(x)$

if and only if $k > 1$

i,e $(x-\lambda) \mid (p(x), p'(x))$ if $k > 1$.

$\Rightarrow$ λ is a ~~m~~ multiple roots (repeated roots) if and only if p and p' have λ as common root.

Therefore if p and p' have no common roots, then all roots of p are simple i.e p has n distinct roots.

Q. let the 3×3 matrix function A be defined on the complex plane by

$$A(z) = \begin{pmatrix} 4z^2 & 1 & -1 \\ -1 & 2z^2 & 0 \\ 3 & 0 & 1 \end{pmatrix}$$

How many distinct values of z are there such that $|z| < 1$ and $A(z)$ is not invertible?

Soln: 4 Ans

Given

$$A(z) = \begin{pmatrix} 4z^2 & 1 & -1 \\ -1 & 2z^2 & 0 \\ 3 & 0 & 1 \end{pmatrix}$$

$$\det(A(z)) = 8z^4 + 6z^2 + 1$$

$$f(z) = 8z^4$$

$$g(z) = 6z^2 + 1$$

By Rouche's theorem, f and $f+g$ have 4 zeroes with $|z| < 1$.

$\Rightarrow$ det(A(z)) has four zeros in the unit disc.

Also $\frac{d}{dz}$ (det (A(z)) = $\cancel{z}$ ($\cancel{3z}$ 3z.

= ~~det~~ derivative of det (A(z))

$$= z\left(32z^3+12\right)$$

$$\Rightarrow z = 0, \pm i\sqrt{\frac{3}{8}}$$

which are not zeroes of det(A(z))

$\Rightarrow$ $\frac{d}{dz}$(det(A(z))) and det(A(z)) have no common roots.

So all roots of det (A(z)) are simple

i.e all the four zeros are simple, so they are distinct.

Q. If the function f is analytic in the entire complex plane, and if f maps every unbounded sequence to an unbounded sequence, then f is a polynomial. True / False

Soln. f doesn't have a removable singularity at ∞ because f is unbounded near ∞.

Now if f had an essential singularity at infinity ~~led~~ then for any $w \in \mathbb{C}$ there would exist a sequence $z_n \to \infty$ with $\lim f(z_n) = w$

For example, take ~~f(z)~~ $z_n = 2\pi i n$

$f(z) = e^z$ and $w = 1 \in \mathbb{C}$

then $f(z_n) = e^{2\pi i n}$

$$\lim_{n \to \infty} f(2\pi i n) = \lim_{n \to \infty} e^{2\pi i n} = 1$$

This leads to a contradiction that

f is an unbounded sequence

Therefore, f has a pole at infinity

and f is ~~to~~ a polynomial.

Note :- $\sin z$ is unbounded in $\mathbb{C}$ but

it is bounded in the ~~real~~ real axis,

so it maps unbounded real sequences to

bounded real sequences.

Q. let f be an analytic function on a disc D whose centre is the point z_0. Assume that $|f'(z) - f'(z_0)| < |f'(z_0)|$ on D. Then f is one to one on D.

True / False

Sol: True.

Consider two points z_1 and z_2 in

$$D = \{ |z| < 1 \}.$$

let γ be the line segment connecting z_1 and z_2, parameterized in the usual way.

Since D is convex, γ is contained in D.

$$f(z_2) - f(z_1) = \int_\gamma f'(z)\, dz$$

$$= \int_\gamma f'(z_0)\, dz + \int_\gamma (f'(z) - f'(z_0))\, dz$$

$$= f'(z_0)(z_2 - z_1) + \int_\gamma (f'(z) - f'(z_0))\,dz$$

Here $$\int_\gamma (f'(z) - f'(z_0))\,dz \le \int_\gamma |f'(z) - f'(z_0)|\,|dz|$$

$$= \int_\gamma |f'(z) - f'(z_0)|\,|dz|$$

$$< |f'(z_0)| \int_\gamma |dz|$$

$$= |f'(z_0)|\,|z_2 - z_1|$$

$$\Rightarrow f(z_2) - f(z_1) < |f'(z_0)|\,|z_2 - z_1| + |f'(z_0)|\,|z_2 - z_1|$$

$$\Rightarrow |f(z_2) - f(z_1)| > |f'(z_0)|\,|z_2 - z_1| - |f'(z_0)|\,|z_2 - z_1|$$

$$= 0$$

$\Rightarrow |f(z_2) - f(z_1)| > 0$

$\Rightarrow f(z_1) \neq f(z_2)$

Since this holds for all pairs of points in D

$\Rightarrow f$ is injective.

Q. Let $p(x)$ be a polynomial with real coefficients and with leading coefficient 1. Suppose that $p(0) = -1$ and that $p(x)$ has no complex zero inside the unit circle. Then find the value of $P(1)$?

Sol

let $p(x) = x^n + x^{n-1} + \dots + x - 1.$

$p(0) = -1.$

let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the roots of P.

we have $\prod \lambda_i = (-1)^{n+1}$

i.e the product of all zeros of $p(x)$ equals ± 1.

Therefore if there are no roots inside the unit circle, then there are no roots outside the unit circle either.

Hence, all roots are on the unit circle.

It is given that $p(0) = -1 < 0$

and $\lim_{x \to \infty} p(x) = +\infty$

$\Rightarrow$ By intermediate theorem, $p(x)$ has a real roots (zero) in the interval $(0, \infty)$.

Since domain is unit circle, so $p(1) = 0$

$1 \in (0, \infty)$

Cauchy's root test

Given $\sum a_n$ (series of positive real terms)

put $\alpha = \lim_{n\to\infty} \sup \sqrt[n]{|a_n|}$ then

(1) If $\alpha < 1$ then $\sum a_n$ converges

(2) If $\alpha > 1$ then $\sum a_n$ diverges

(3) If $\alpha = 1$, the test gives no information or test is inconclusive.

Q. For which $z \in \mathbb{C}$ does

$$\sum_{n=0}^{\infty} \left(\frac{z^n}{n!} + \frac{n^2}{z^n} \right) \text{ converge?}$$

Ans

Let $a_n = \frac{z^n}{n!}$, $b_n = \frac{n^2}{z^n}$

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \frac{z^n}{n!} \text{ converges for all } z$$

If $z \neq 0$, then

$$\left| \frac{n^2}{z^n} \right|^{1/n} = \frac{(n^{1/n})^2}{|z|} \to \frac{1}{|z|}$$

By ~~lettt~~ cauchy root test

$$\sum \frac{n^2}{z^n} \text{ converges for } \frac{1}{|z|} = \alpha < 1$$

i.e $\frac{1}{|z|} < 1 \Rightarrow |z| > 1$

and $\sum \frac{n^2}{z^n}$ diverges for $|z| < 1$.

Hence the series $\sum \left(\frac{z^n}{n!} + \frac{n^2}{z^n}\right)$ converges

for $|z| > 1$, and diverges for $|z| < 1$.

The series is undefined for $z = 0$.

Let $|z| = 1$. The sequence $\frac{z^n}{n!}$ tends to

zero and the sequence $\frac{n^2}{z^n}$ tends to

∞.

Therefore the sequence $\left(\frac{z^n}{n!} + \frac{n^2}{z^n}\right)$ tends to ∞,

which implies that the series

$$\sum \left(\frac{z^n}{n!} + \frac{n^2}{z^n}\right) \text{ is divergent for}$$

$|z| = 1$.

~~Therefore~~ $\cancel{\sum_{n=0}^{\infty} \frac{z^n}{n!}}$ ~~converges~~ +

Therefore $\sum_{n=0}^{\infty} \left(\frac{z^n}{n!} + \frac{n^2}{z^n}\right)$ converges

exactly when $|z| > 1$