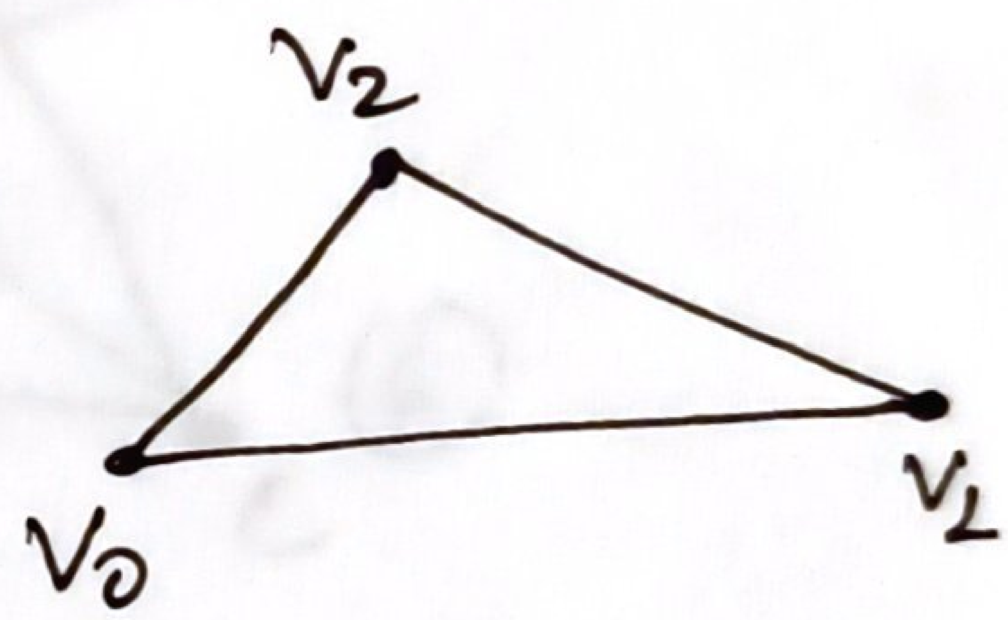
Simplex :- A simplex is a generalization of the notion of a triangle or tetrahedron to ~~arbit~~ arbitrary dimension.

Notion (idea)

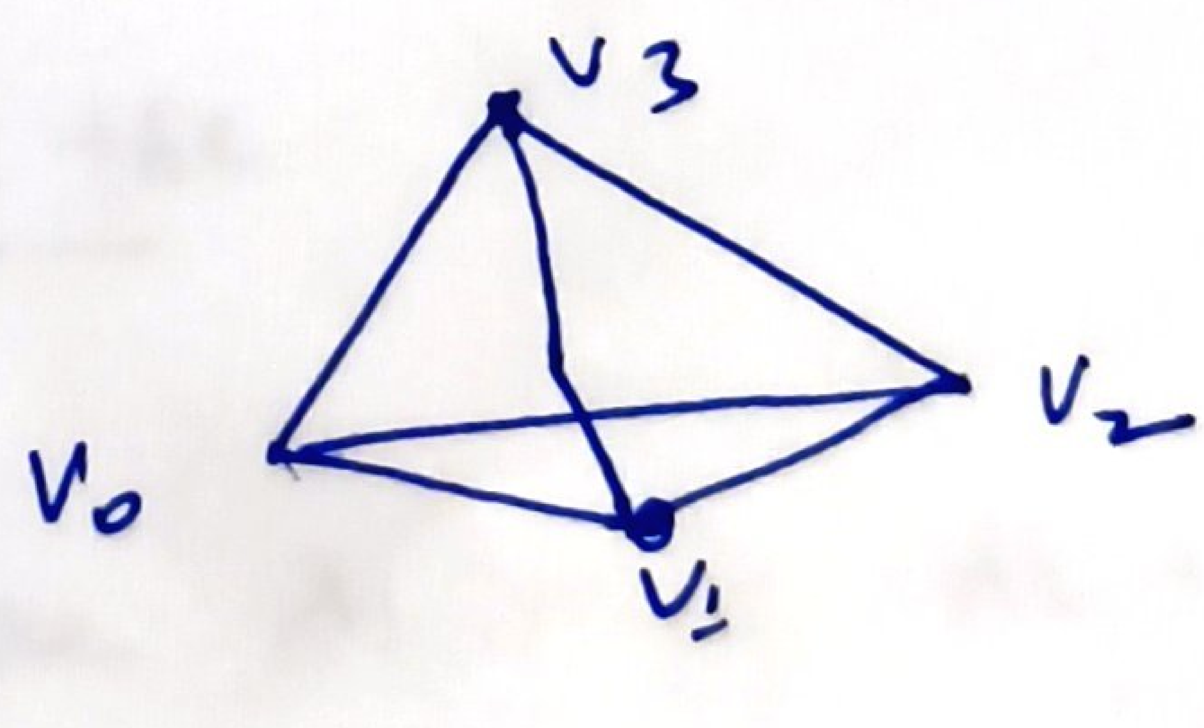
0 - simplex = point

1 - simplex = closed line segment

2 simplex = triangle



3 simplex = tetrahedron (solid)



Convex Hull :-

The convex hull of a set of points S in n dimensions is the intersection of all convex sets containing S.

For N points $P_1, P_2, \ldots, P_N$, the convex hull C is then given by the expression

$$C = \left\{ \sum_{i=1}^{N} \lambda_i P_i : \lambda_i \geq 0 \text{ for all } i \text{ and } \sum_{i=1}^{N} \lambda_i = 1 \right\}$$

Also, we can say that

Given points $x_0, \ldots x_p \in R^n$, the convex hull of the set $S = \{x_0, x_1, \ldots, x_p\}$ is the smallest convex set in R^n that contain S.

Convex hull is given as

$$\{ t_0 x_0 + \dots + t_p x_p \in \mathbb{R}^n : \sum t_i = 1, t_i \geq 0 \}$$

$$\Rightarrow \left\{ \sum_{i=0}^{n} t_i x_i \mid t_1, t_2 \dots, t_n \in \mathbb{R}_{\geq 0} \text{ and } \sum_{i=1}^{n} t_i = 1 \right\}$$

Note $p = n \geq 0$ is an integer.

Flat : (~~Eucl~~ Euclidean subspace) :

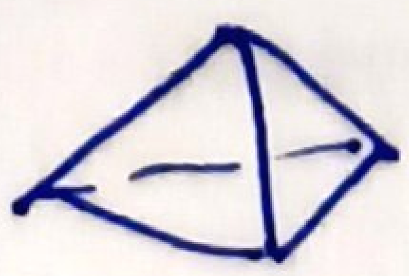
A Flat or Euclidean Subspace is a subset of a euclidean space that is itself a euclidean space of lower dimension.

Example :- Flats in two dimensional space are points and lines, and the flat in 3-dimensional spaces are points, lines and planes.

Note :- In a n- dimensional space, there are flats of every dimension from 0 to n-1, flats of dimension n-1 are called hyperplanes.

~~Polyhedron~~ / Polyhedron :-

A polyhedron is a three-dimensional shape with a polygonal faces, straight edges and sharp corner or vertices.

Eg:  → ~~to~~ tetrahedron

Polytope :-

A polytope is a geometric object with flat sides (faces).

<u>Example</u> :- two dimensional polygon is a 2-polytope and a three-dimensional polyhedron is a 3-polytope

<u>Note</u> :- An n-polytope P is the convex hull of finitely many ~~pair~~ points in $\mathbb{R}^n$.

Fig. 1

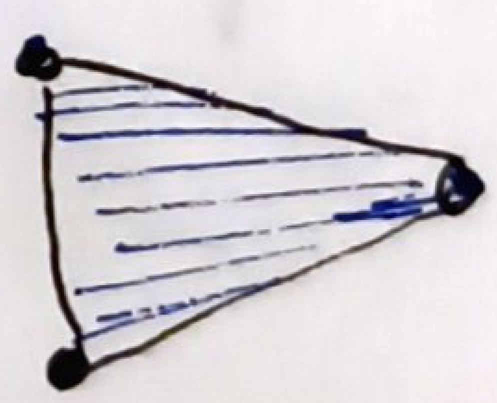
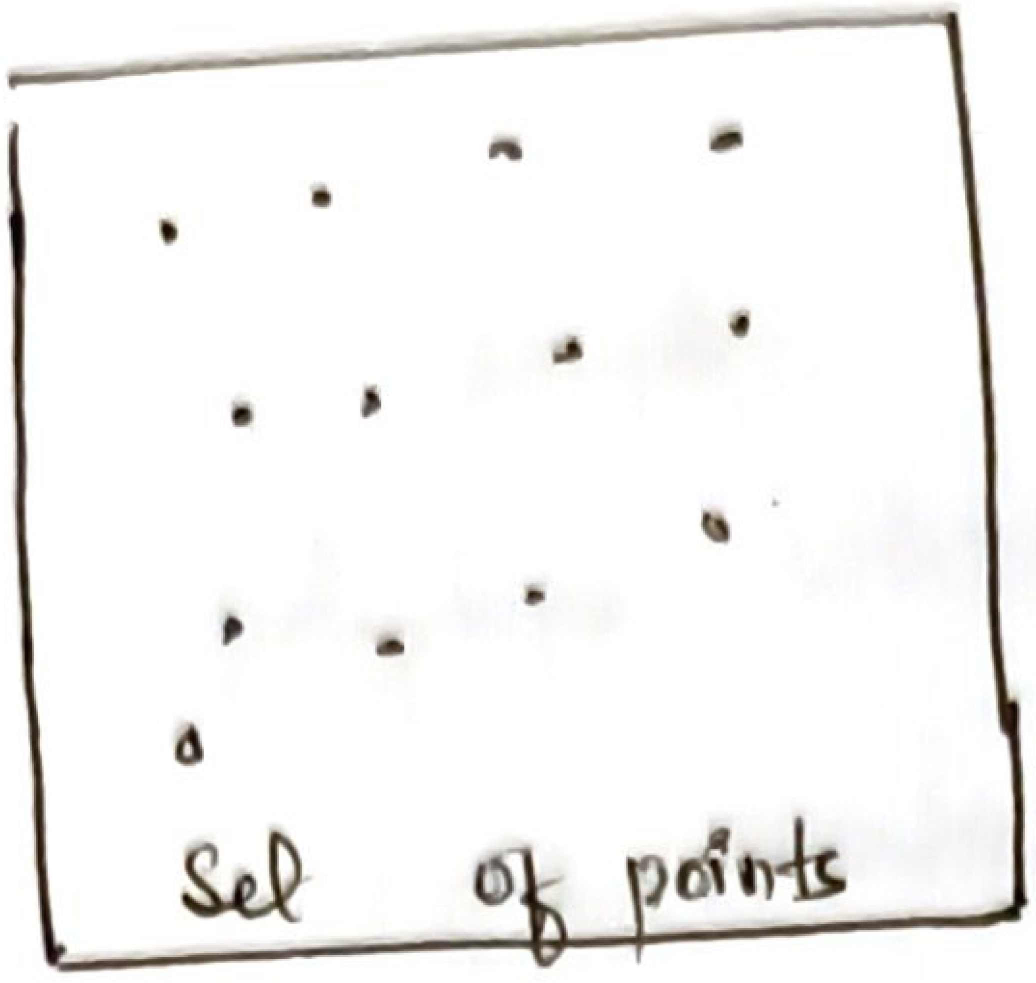
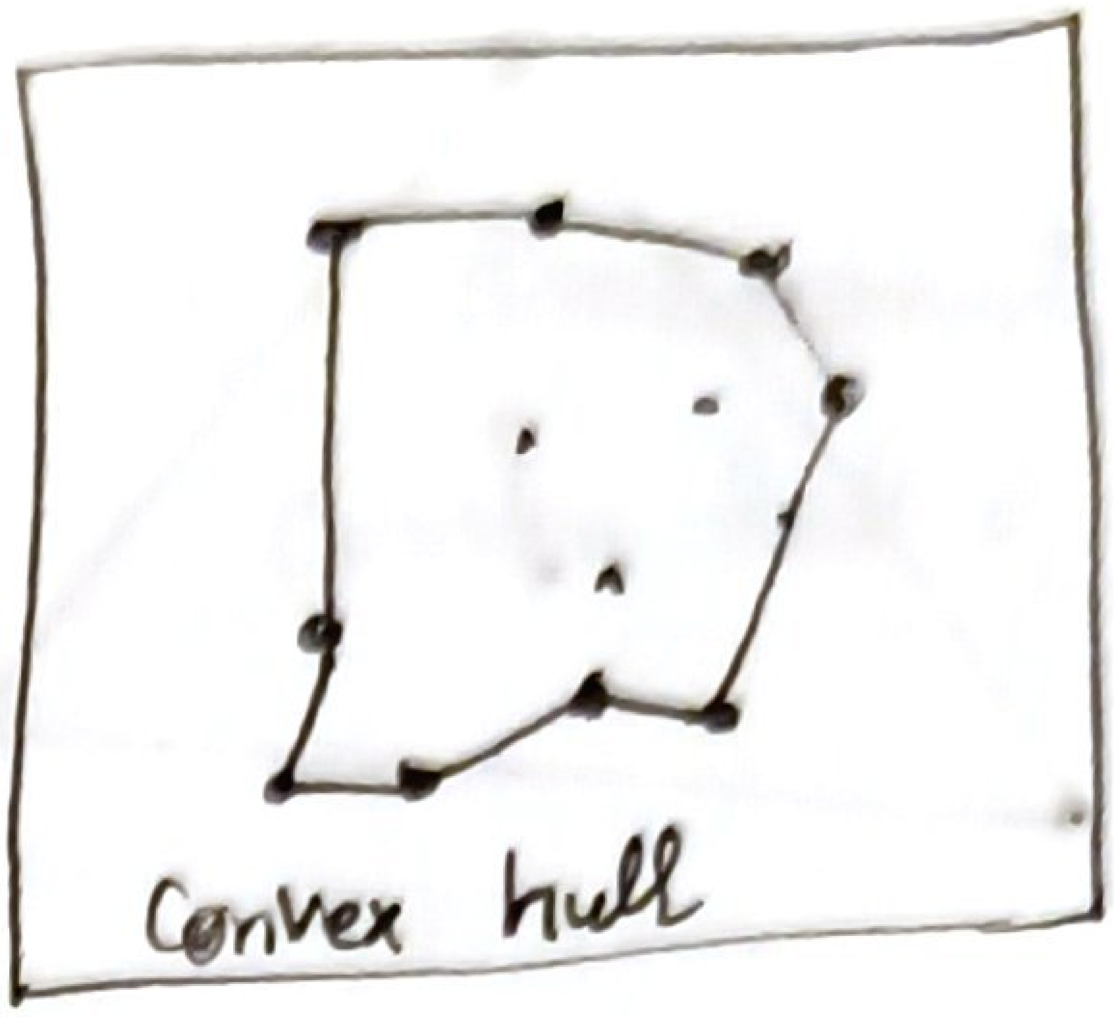


Fig 2.

The shaded triangle in fig 2 is the convex hull of three black points.





~~Con~~ Convex combination :

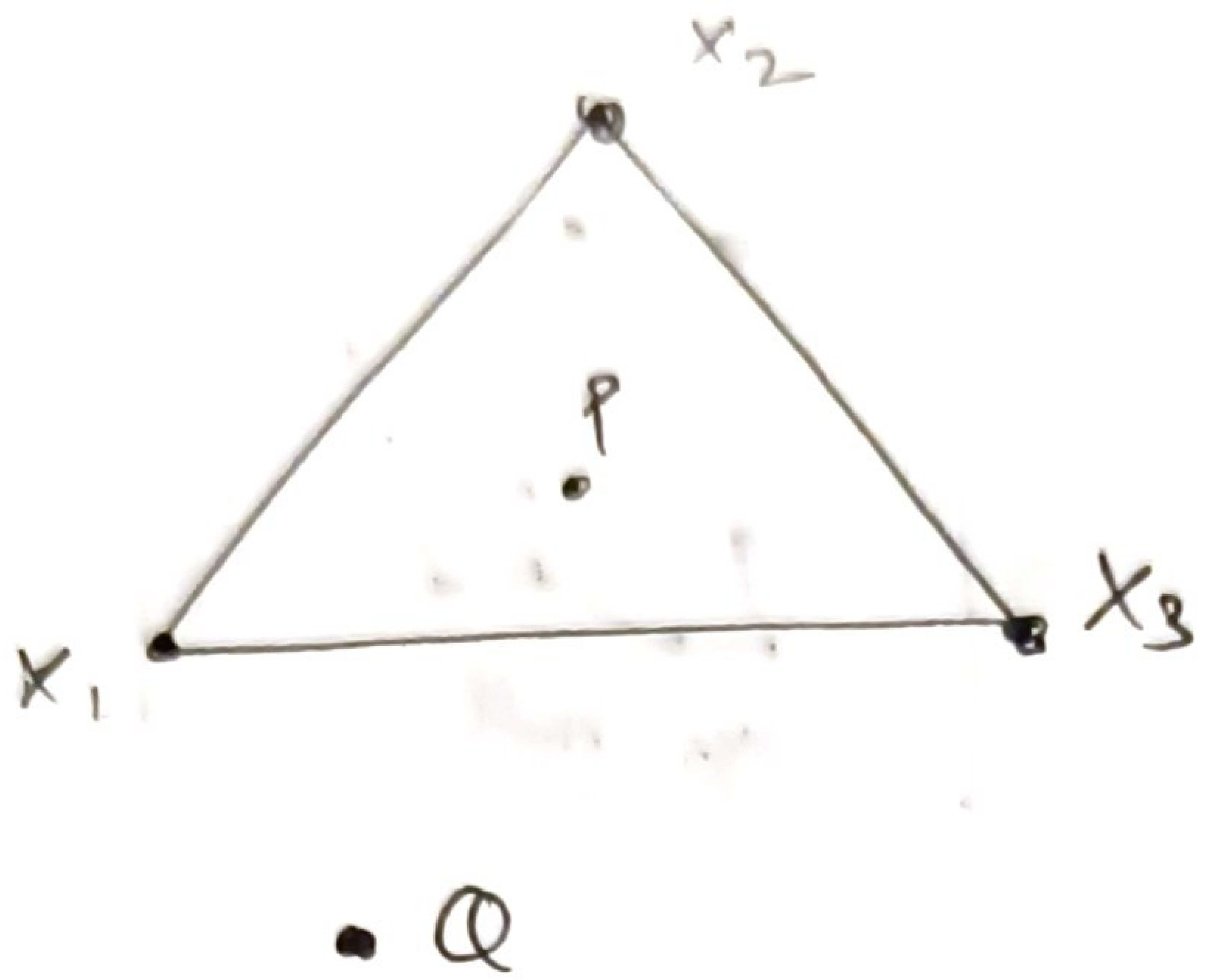
Given a finite number of points $x_1, x_2, \dots, x_n$ in a real vector space, a convex combination of these points of the forms

$$\alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n$$

where the real number α_i satisfy $\alpha_i \geq 0$

and $\alpha_1 + \alpha_2 + \alpha_3 + \dots + \alpha_n = 1$.

Note: A subset A of a vector space V is said to be convex if $\lambda x + (1-\lambda) y$ for all vector $x, y \in A$ and for all scalars $\lambda \in [0, 1]$.



Given three points x_1, x_2, x_3 in a plane as shown in the figure, the point P is a convex combination of the three points, while Q is not.

Convex hull :-

A set of points in a Euclidean space is defined to be convex if it contains the line segments connecting each pair of its point.

The convex hull of a given set X may be defined as

1. the intersection of all convex sets containing X.
2. The set of all convex combinations of points in X.

n- simplex :-

A n- simplex is a n- dimensional polytope which is the convex hull of its $n+1$ vertices.

Also, An n-simplex $[v_0, \ldots, v_n]$ is the convex hull of $n+1$ ordered points (called vertices) in $\mathbb{R}^m$ for which $v_1 - v_0, \ldots, v_n - v_0$ are linearly independent.

The standard n- simplex is

$$\Delta^n = \left\{ (t_0, t_1, \ldots, t_n) \in \mathbb{R}^{n+1} \;\middle|\; \sum_{i=0}^{n} t_i = 1 \text{ and } t_i \geq 0 \; \forall i \right\}$$

and there is a canonical map

$$\Delta^n \longrightarrow [v_0, \ldots, v_n] \quad \text{via}$$

$$(t_0, t_1, \ldots, t_n) \longrightarrow \sum_i t_i v_i$$

called barycentric coordinates on $[v_0, \ldots, v_n]$

let A be p-simplex.

If A is the convex hull of $\{x_0, \ldots, x_n\}$ then every point of A has a distinct unique representation in the form

$\sum t_i x_i$ where $t_i \geq 0$ forall i and

$\sum t_i = 1$.

The points x_i are called the vertices of A.
If the vertices of A have been given a specific order, then A is called an ordered simplex.

let σ_p be the set of all points $(t_0, \ldots, t_n)$ $\in \mathbb{R}^{n+1}$ with $\sum t_i = 1$ and $t_i \geq 0$ forall i.

Note: ① t_i are called bary-centric coordinates.

② we can take $p = n$ i.e. $\sigma_p = \sigma_n$

Then σ_p is a p-simplex with vertices

$$x_0 = (1, 0, \ldots 0)$$

$$x_1 = (0, 1, 0 \ldots 0)$$

$$\vdots$$

$$x_i = (0, \ldots, 1, 0 \ldots)$$

$$\vdots$$

$$x_p = (0, \ldots, 0, 1) = x_p = x_n.$$

i.e. σ_p is called the standard p-simplex with natural ordering.

we can write

$$\boxed{\sigma_p = \Delta^p}$$

Now, let $A \subset \mathbb{R}^k$ be an ordered simplex with vertices $x_0, \ldots, x_n$. If we define

$$f : \sigma_p \longrightarrow A \text{ by}$$

$$f(t_0, t_1, \ldots, t_p) = \sum t_i x_i \text{, then}$$

f is continuous.

Since $\sigma p = \Delta^p$ is compact and ~~is~~ A is hausdorff

Since A is hausdorff because we can get two subsimplex P_1 - simplex $= A_1$ and P_2 - simplex $= A_2$ such that

$$A_1 \cap A_2 = \phi$$

$\Rightarrow$ ~~the~~ there exist ~~a~~ disjoint ~~open~~ open subsets U and V with $x \in A_1$ in U and ~~A_2 in V~~. $y \in A_2$ in V.

~~$\Rightarrow$ f is~~

Therefore f is a homeomorphism

by using the theorem

let X and Y be topological spaces and let $f: X \longrightarrow Y$. If X is compact, Y is Hausdorff, f is continuous $\Rightarrow f$ is a homeomorphism.

Note:

$\sigma = [v_0, \ldots, v_n] : \Delta^n \longrightarrow \mathbb{R}^N$ defined by

$$t \longmapsto \sum_i t_i v_i \quad \text{is injective.}$$

σ is called an n-simplex and the point $v_i = \sigma(e_i)$ are called its vertices.

we identify σ with its image (convex hull)

$$\Delta(\sigma) = \mathrm{Con}(v_0, \ldots, v_n) = \text{Convex hull}(v_0, \ldots v_n)$$

$$= \left\{ \sum_i t_i v_i \,\middle|\, \sum t_i = 1,\ t_i \geq 0\ \forall i \right\} \subset \mathbb{R}^N$$

equipped with an ordering of vertices.

$\Delta^n = [e_0, \ldots, e_n] \subset \mathbb{R}^n$, where $(e_0, \ldots, e_n)$ is the standard bases of $\mathbb{R}^{n+1}$

Δ^n :- Standard n-simplex.

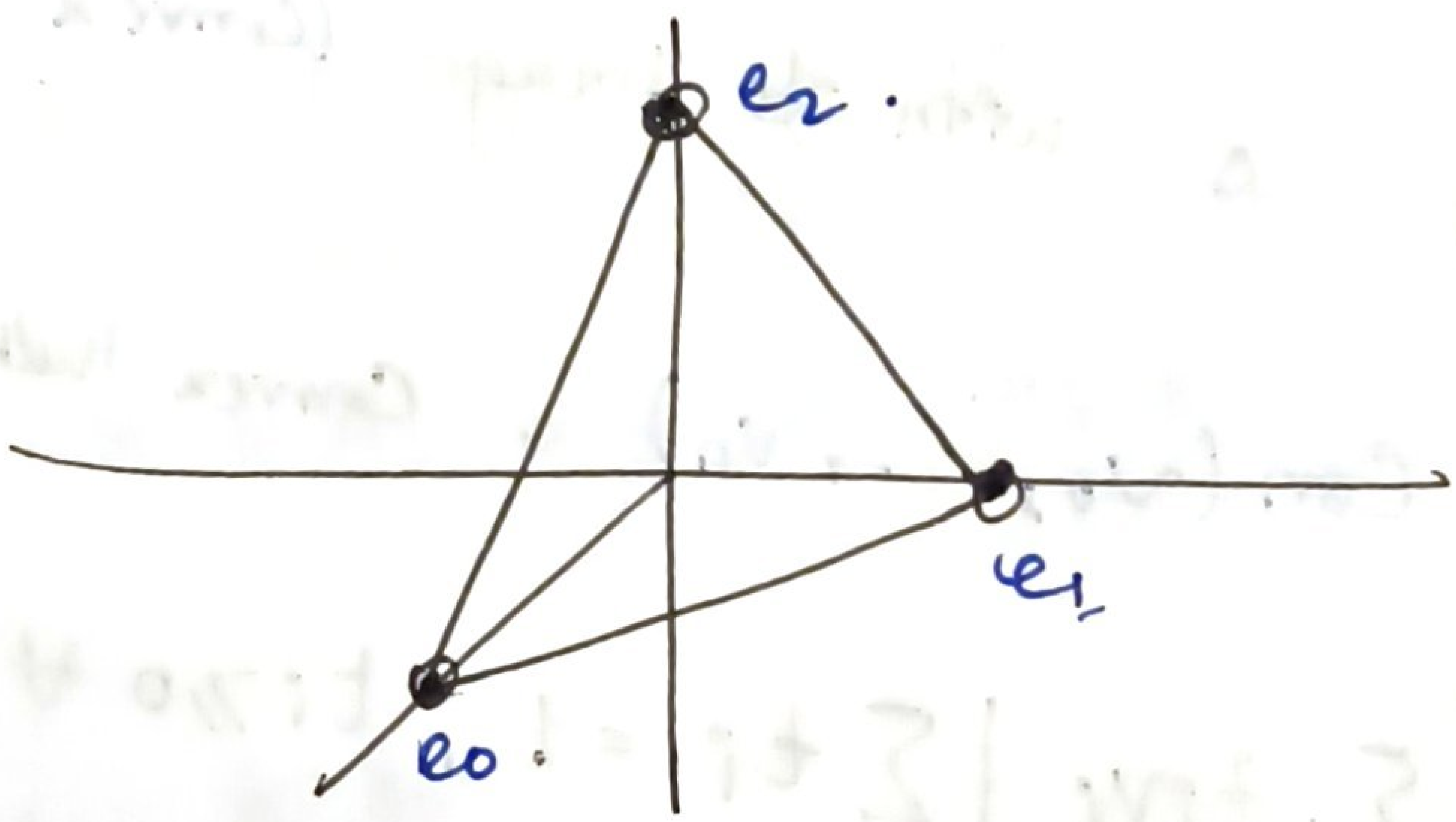
→ The standard (ordered) n-simplex $[v_0, \quad, v_n]$

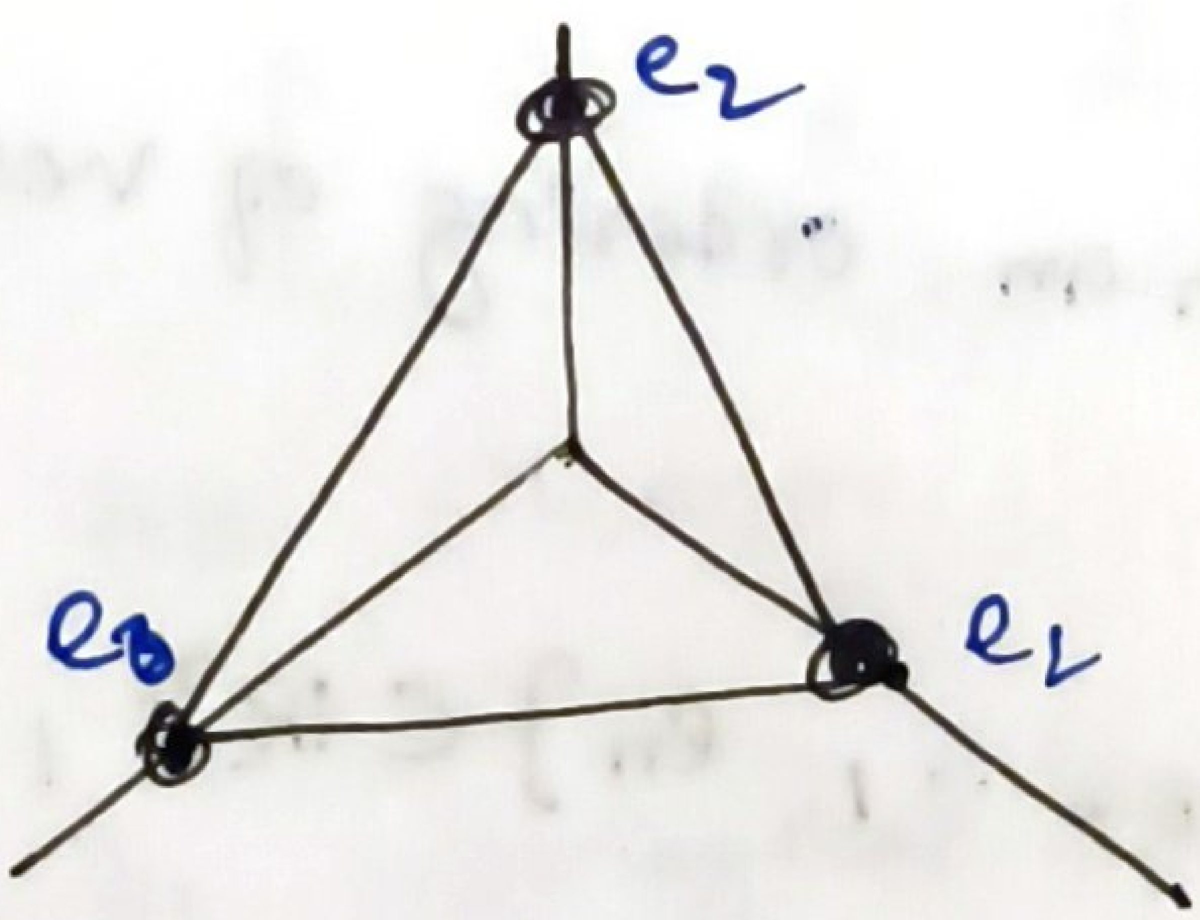
is the convex hull of the set ~~set~~.

$\{e_1, \dots, e_{n+1}\}$ where $e_i = (0, \quad, 1, \dots 0)$

$\in \mathbb{R}^{n+1}$.

<u>Example</u>: Take $n = 2$





let $V_0 = 0$ and $v_i = e_i$ for

$i = 1, \dots, n+1$. Then

$$\Delta^n = \text{convex-hull}\left(\{v_1 - v_0, \dots, v_{n+1} - v_0\}\right)$$

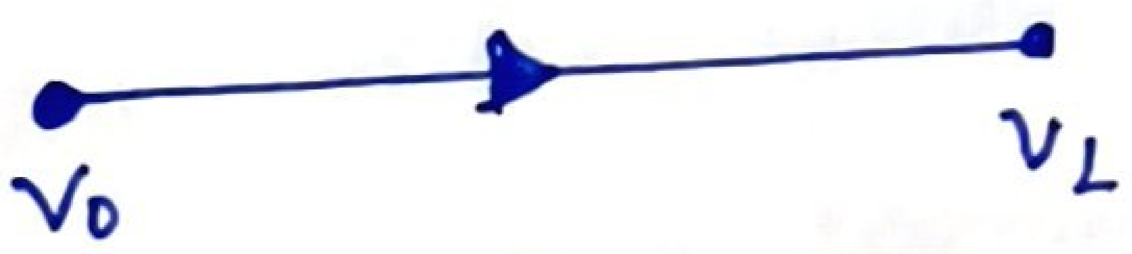
$$= \sum_{i=1}^{n} t_i e_i \in \mathbb{R}^{n+1} \;\Big|\; t_i \geq 0, \sum_{i=1}^{n} t_i = 1 \Big\}$$

Ex

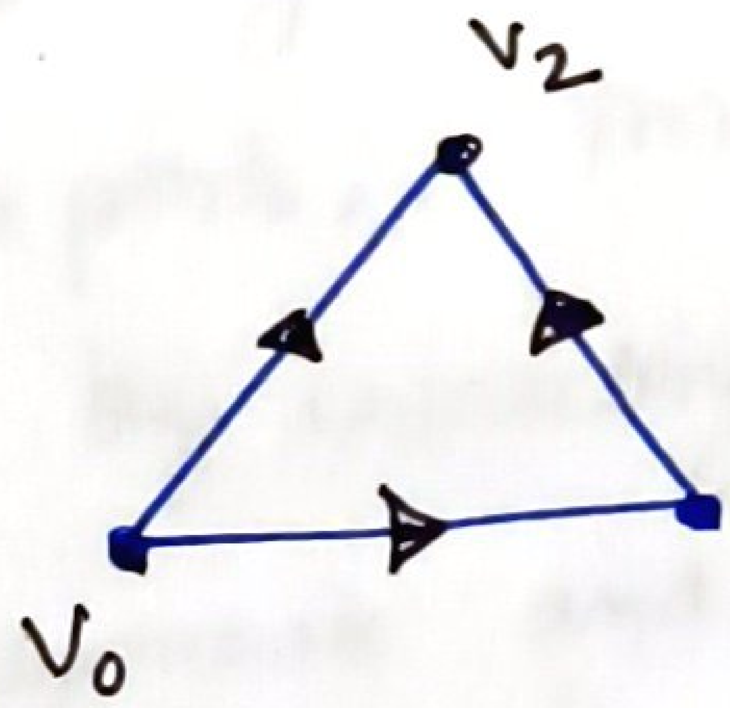
Example:-

$\Delta^0 = \bullet\, v_0$

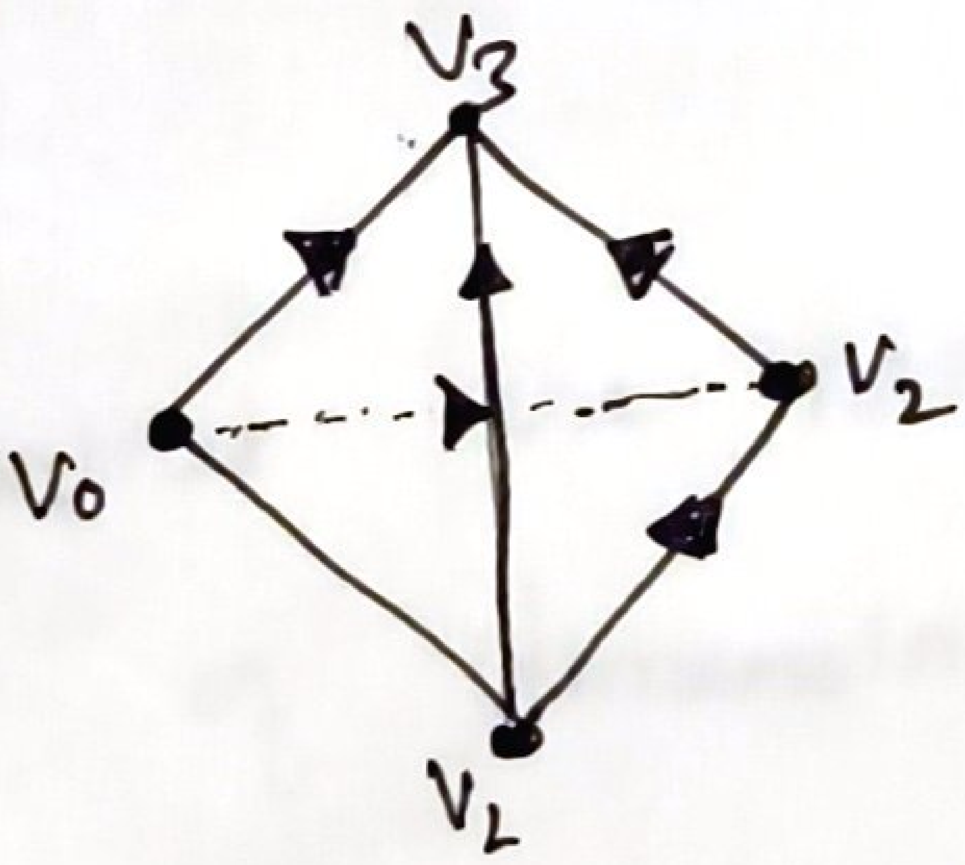
$\Delta^1 =$



$\Delta^2 =$



$\Delta^3 =$



An standard (ordered) n-simplex has

$n+1$ face map for each $i = 0, \dots n$

$[v_0 \dots \hat{v_i} \dots v_n]$ = an ordered $(n-1)$ simplex

Polyhedron :

It is note that a simplical complex is a set of simplexes, not points of $\mathbb{R}^n$.

But if one takes the union of all the points lying on the simplexes of K i.e

$$|K| = \bigcup_{\sigma \in K} \sigma$$, then $|K|$ is a subspace of the Euclidean space $\mathbb{R}^n$.

This space $|K|$ is called polyhedron of K.

→ The polyhedron defined by K is the union of the simploces in K, and denoted by $|K|$.

The dimension of a simplicial complex K, denoted dim K, is the largest dimension of a simplex in K.

i.e dimension of K is the highest dimension of a simplex of K.

~~Def~~ d - skeleton of K

Given a simplicial complex K and a non-negative integer d, the d-skeleton of K, denoted $K^{(d)}$ or $K_{(d)}$ is the set of simplices in K of dimension no greater than d.

$\Rightarrow$ The d-skeleton $K(d)$ of a simplicial complex K is the sub simplicial complex given by the simplices of K of dimension at most d

Also, we can ~~say~~ say that the d-skeleton $K^{(d)}$ of K is the union of the n-simplices in K for $n \leq d$.

d- skeleton K^d is the collection
$$\{ \sigma \in K \mid \dim(\sigma) \leq d \}$$

Example :- 0 - Skeleton is the vertices,
1 - skeleton is the set of vertices and edges

2 - skeleton is the set of vertices, edges and triangles and so on.

A polygonal curve is a finite chain of line segments.

⇒ line segments called edges, ~~ther~~ their endpoints called vertices.

A simple polygon is a closed polygonal curve without self - intersection.

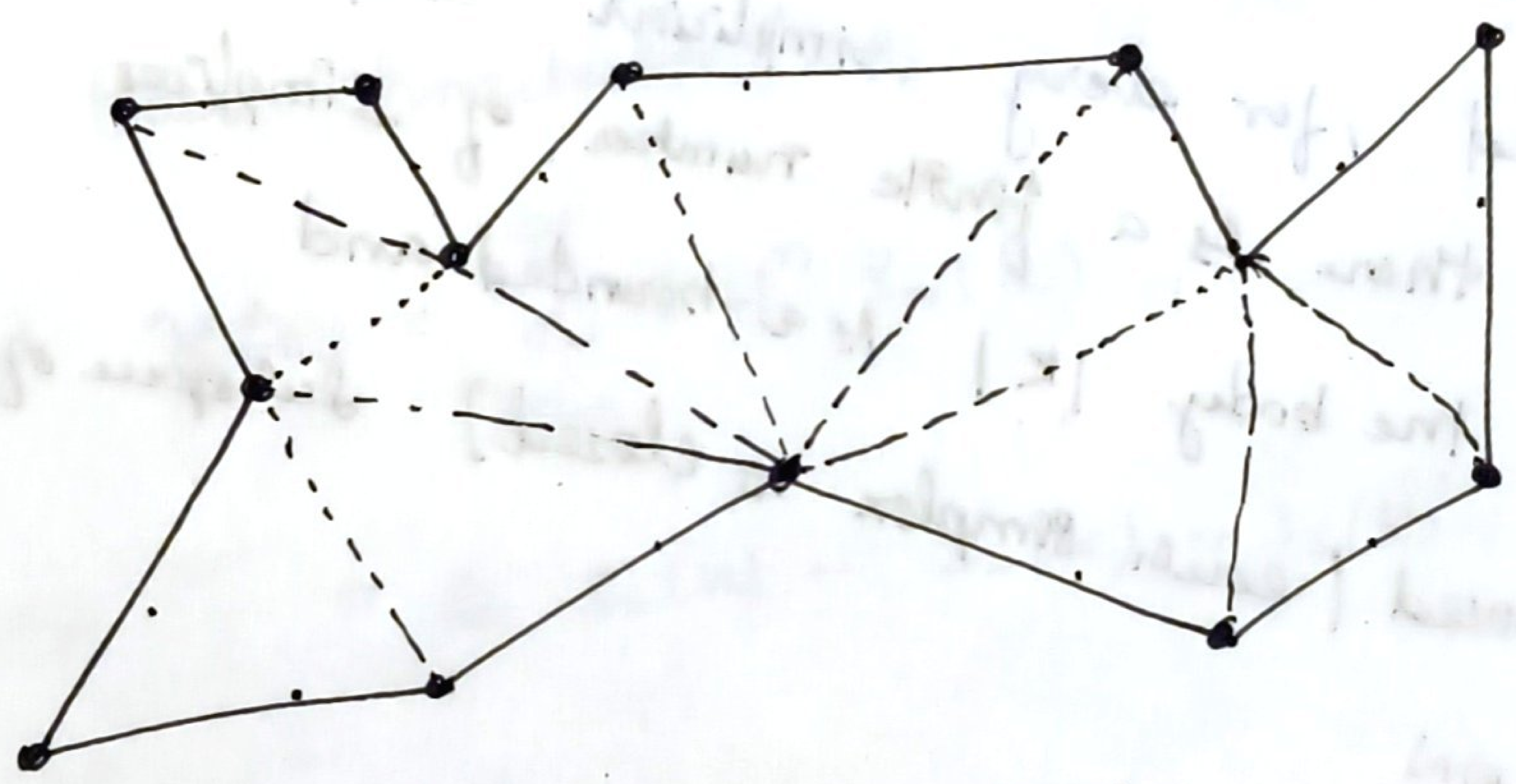
TRIANGULATION :-

Triangulation is the division of a surface or plane polygon into a set of triangles, usually with the restriction that each triangle side is entirely shared by two adjacent triangles.

Triangulation reduces complex shapes to collection of simpler shapes.

Every simple polygon admits a triangulation.

⇒ Every triangulation of an n-gon has exactly $n-2$ triangles



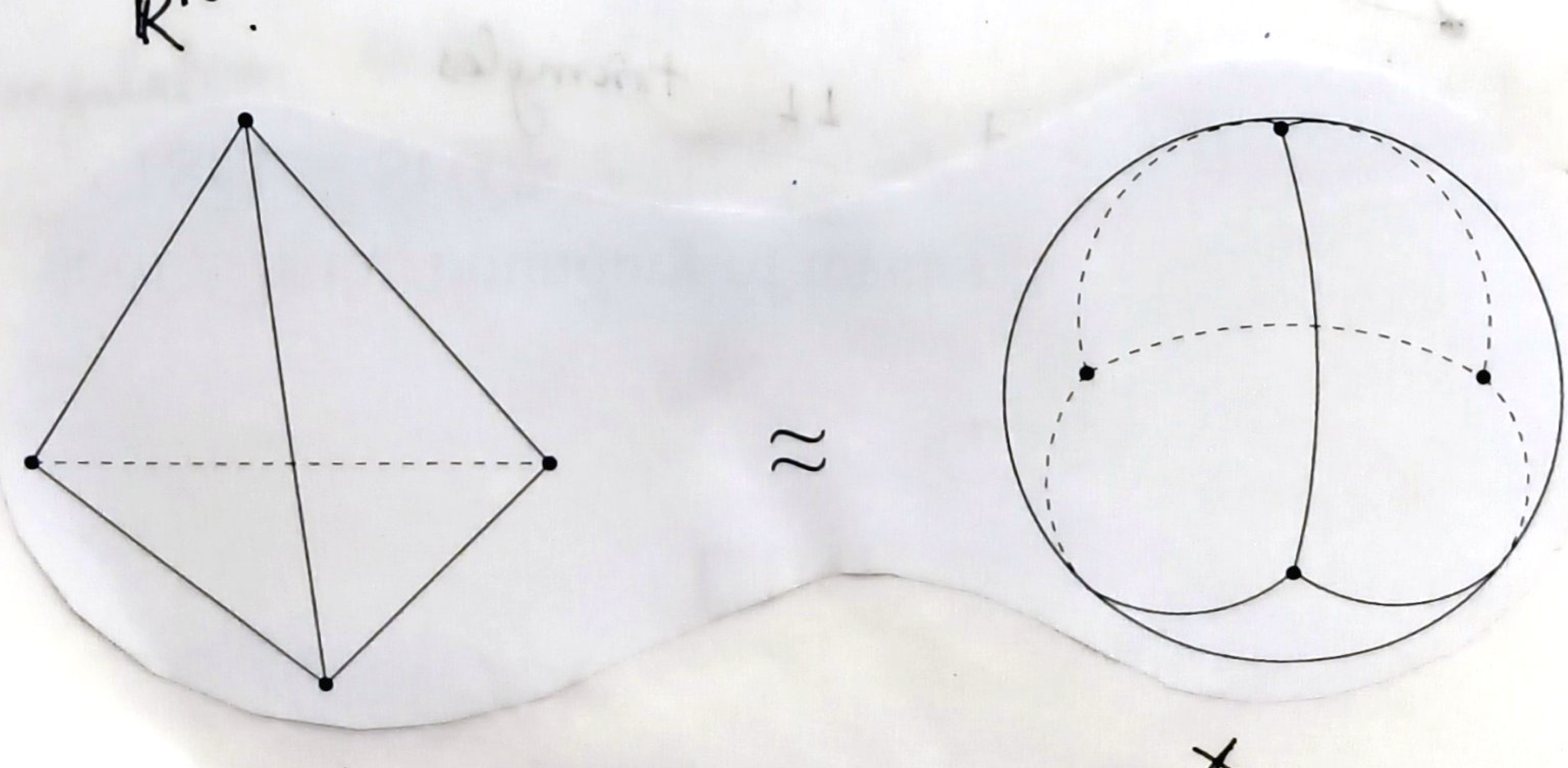
$n = 13$ and 11 triangles

A triangulation of a topological space X is a homeomorphism $h : |K| \longrightarrow X$, where K is some simplicial complex.

<u>Note:-</u>

If a space X admits a triangulation, then X is compact.

Indeed, for every simplicial complex K, if there is a finite number of simplices then the body $|K|$ is a bounded and closed (each simplex is closed) subspace of R^N.



$|K|$ $\approx$ X

Euler Characteristics :

Suppose K is a simplicial complex.

$$S_n(K) = \# \{ \text{simplices } \sigma \in K \mid \dim \sigma = n \}$$

$S_n(K)$ denote the number of n-simplices of K.

The Euler characteristic of K is the integer

$$\chi(K) \ \cancel{\chi(K)} = \sum_{n=0}^{\infty} (-1)^n S_n(K)$$

$$= \cancel{s_0} \ S_0(K) - S_1(K) + S_2(K) + \cdots$$

Note: If X is a triangulated space, i.e. there is ~~is~~ a triangulation $X \cong |K|$ for some ~~simplicial~~ simplicial complex K, we set

$$\boxed{\chi(X) = \chi(K)}$$

and call $\chi(X)$ the Euler characteristic of the topological space X.

Theorem :-

If K, L are finite simplicial complexes such that $|K|$ is homeomorphic to $|L|$ then $\chi|K| = \chi|L|$

Example :- (1) $X = \{x_0\}$ (one point space)

Then $K_1 = \{[a_0]\}$

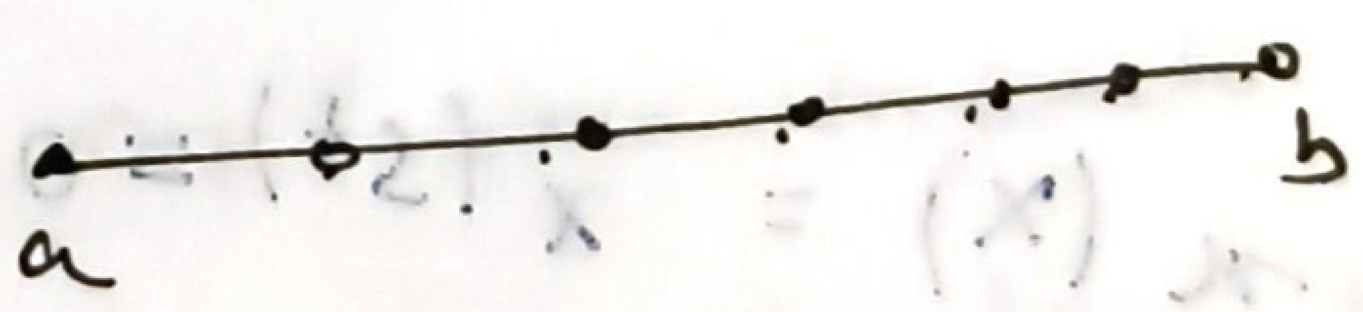
$S_0(K) = 1$, $\chi(X) = 1$.

(2) ~~$K_2 =$~~ ~~$\{a_0, \ldots$~~ $K_2 = [a_0 a_1]$

$S_0 = 2$, $S_1 = 1$

$\chi = 2 - 1 = 1$

$X = [a, b]$



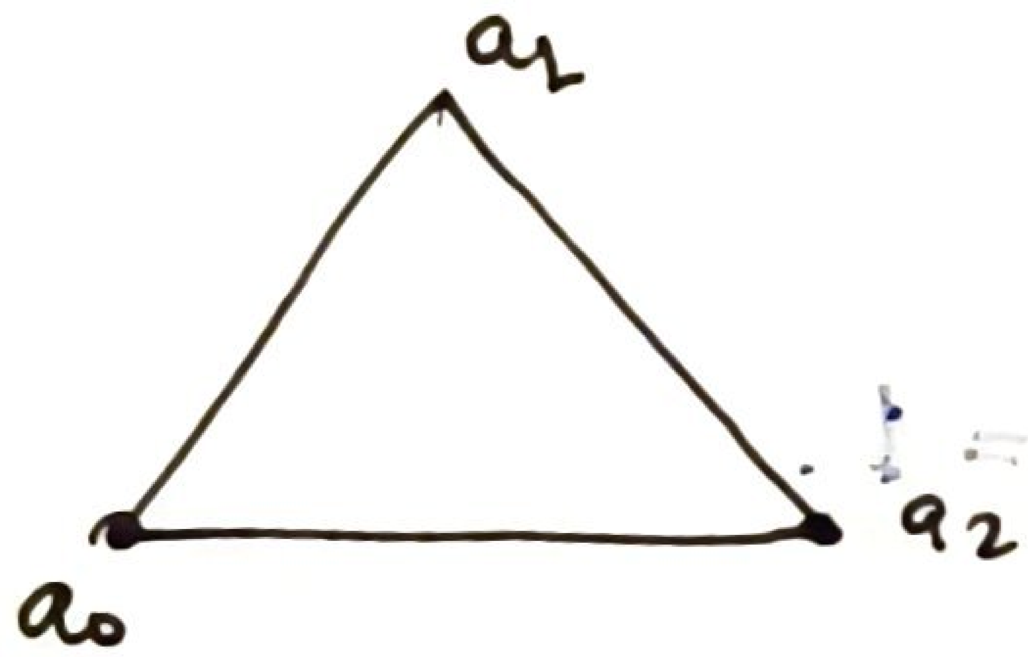
$C_0 = 7$, $C_1 = 6$

$\chi = 7 - 6 = 1$

$$\Rightarrow \chi[X] = \chi[K_2]$$

$$\chi[a,b] = \chi[K_2]$$

(3) $K_3 = [a_0 a_1 a_2]$



$S_0 = 3$

$S_1 = 3$

$$\chi(K) = S_0 - S_1 = 3 - 3 = 0$$

$X = S^1$

$S_0 = 3$

$S_1 = 3$

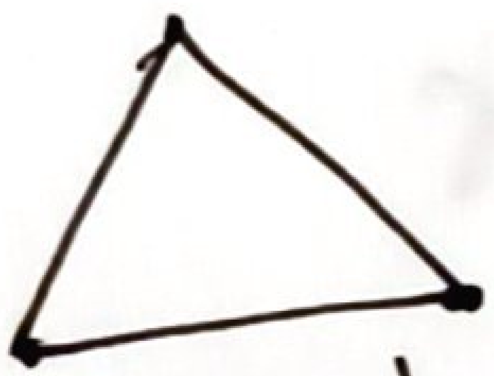
$$\chi(X) = 3 - 3 = 0$$

$$\chi(X) = \chi(S^1) = 0$$

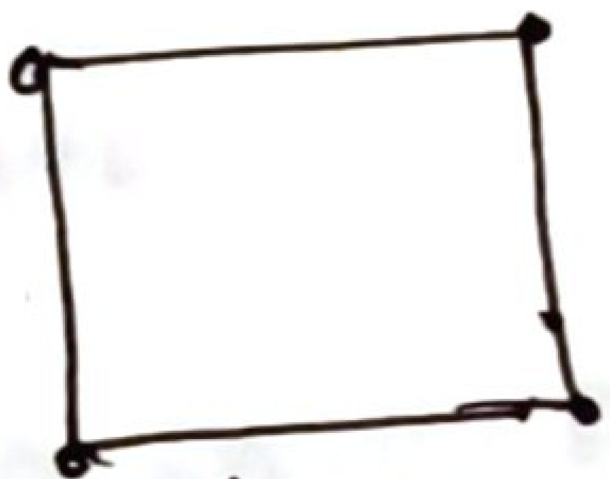
$$\Rightarrow \chi(S^1) = \cancel{\chi(K_3)}\ \chi(K_3)$$

$\Rightarrow |K_3|$ is homeomorphic to the circle S^1.

In each complex K_3 the number of 0-simplices is the same as the number of 1-simplices i.e $S_0 = S_1$.



$|K_3|$



$|K_4|$

$|K_5|$

$|K_6|$

and ~~so~~ ~~so~~ so on ....

$\Rightarrow$ Each space $|K_n|$ is homeomorphic to the circle S^1 for $n \geq 3$ since in each complex K_n the number of 0-simplices is the same as the number of 1-simplices, we have

$\chi(K_n) = 0$ for any $n \geq 3$.

The standard n-simplex $\Delta^n \subset \mathbb{R}^{n+1}$ is the simplex spanned by the basis vector $e_1, e_2, \ldots, e_{n+1}$. It along with all its faces, defines a simplicial complex.

The simplicial $(n-1)$ sphere is the simplicial complex given by the proper face of Δ^n, and all their faces. Its polyhedron is $\partial\Delta^n \subset \mathbb{R}^{n+1}$.

The boundary of $\partial\Delta^n$ is homeomorphic to S^{n-1}.

In $\mathbb{R}^{n+1}$ consider the simplices $\langle \pm e_1, \pm e_2, \ldots, \pm e_{n+1} \rangle$ for each possible combination of signs. So we have ~~to~~ 2^{n+1} simplices in total. Then their union defines a simplicial complex K i.e. K is the simplicial complex given by these simplices and their faces and $|K| \cong S^n$

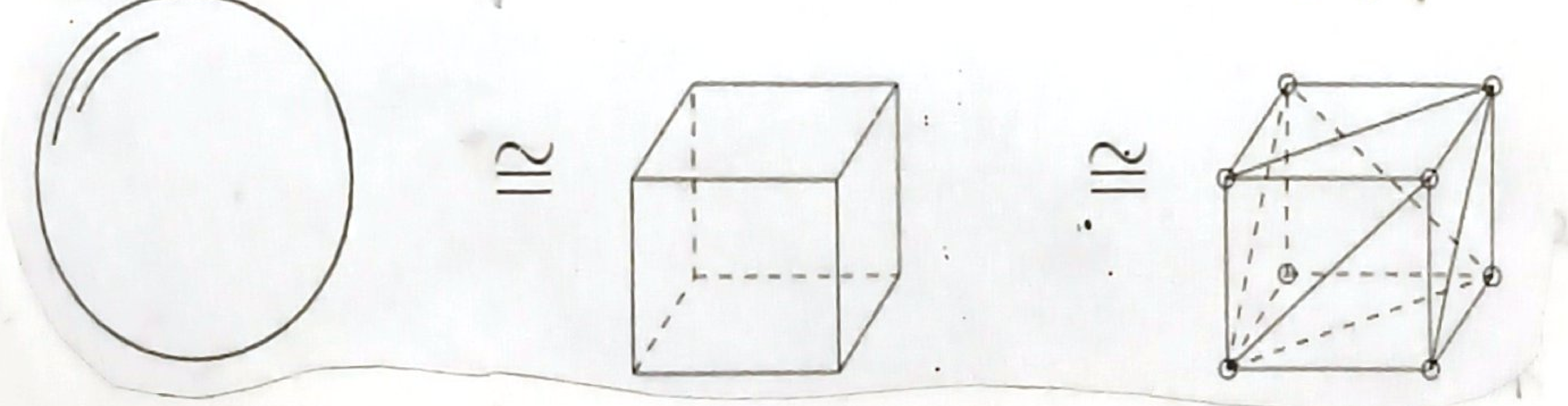
$h : |K| \longrightarrow S^n$ defined by

$f(x)\ h(x) = \dfrac{x}{|x|}$ is a homeomorphism.

Since $|K| \subset \mathbb{R}^{n+1}$ and h is the restriction of the map $\mathbb{R}^{n+1} \setminus \{0\} \longrightarrow S^n$

Thus define a triangulation of S^n

S^2 $\qquad\qquad\qquad\qquad\qquad\qquad$ $|K|$



$\downarrow$ $\qquad\qquad\qquad\qquad$ $\downarrow$

$\chi(S^2) = 2$

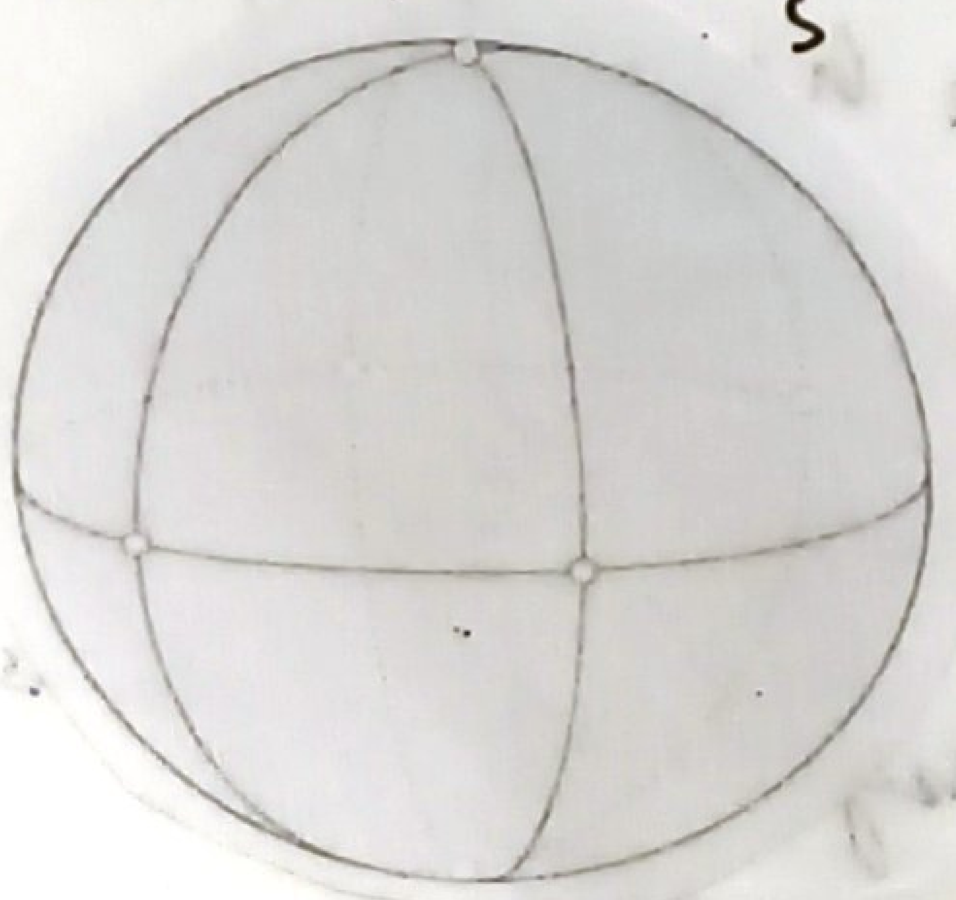
$s_0 = 8$

$s_1 = 18$

$s_2 = 12$

$\chi(K) = 8 - 18 + 12 = 2$

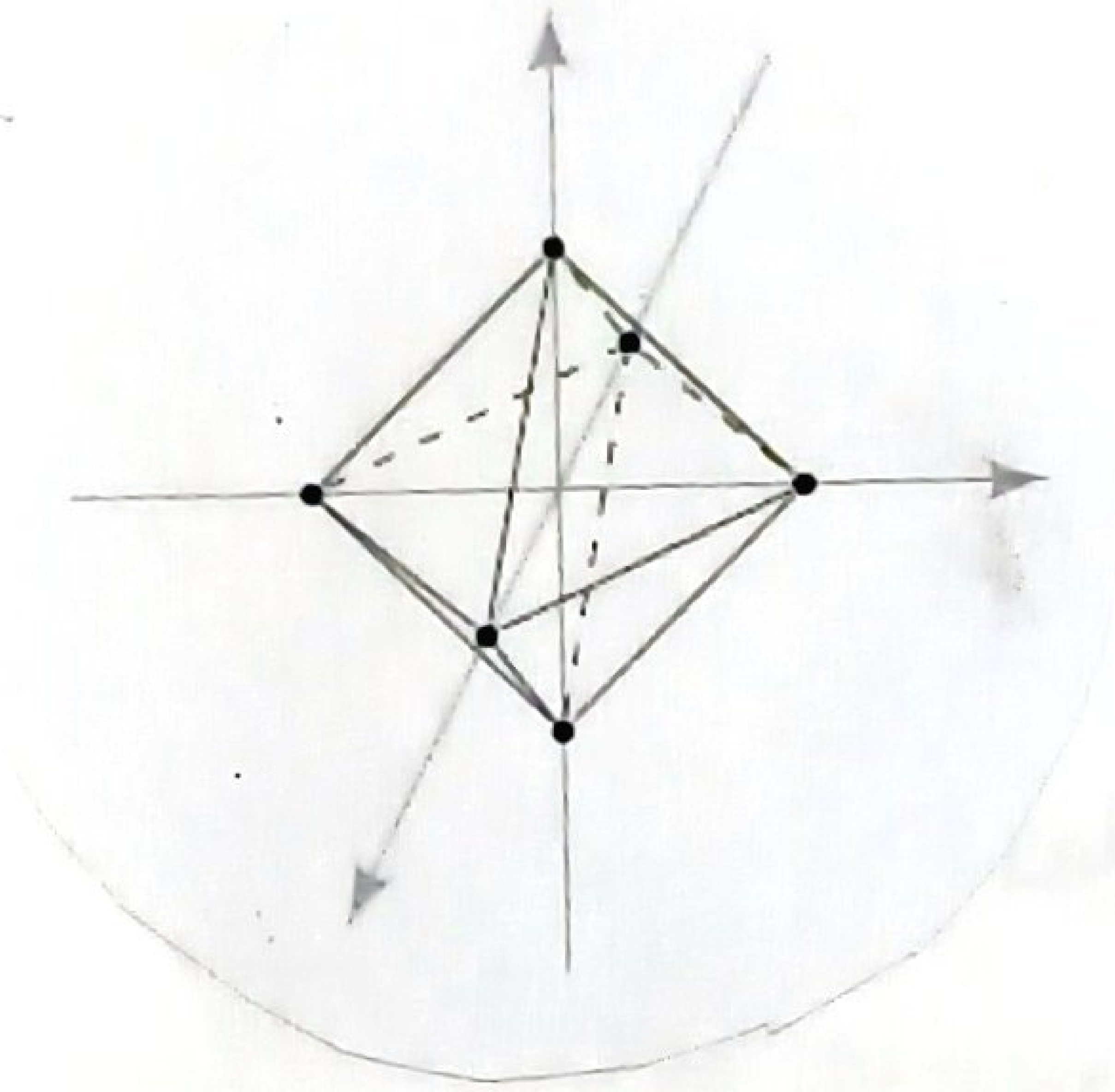
S^2



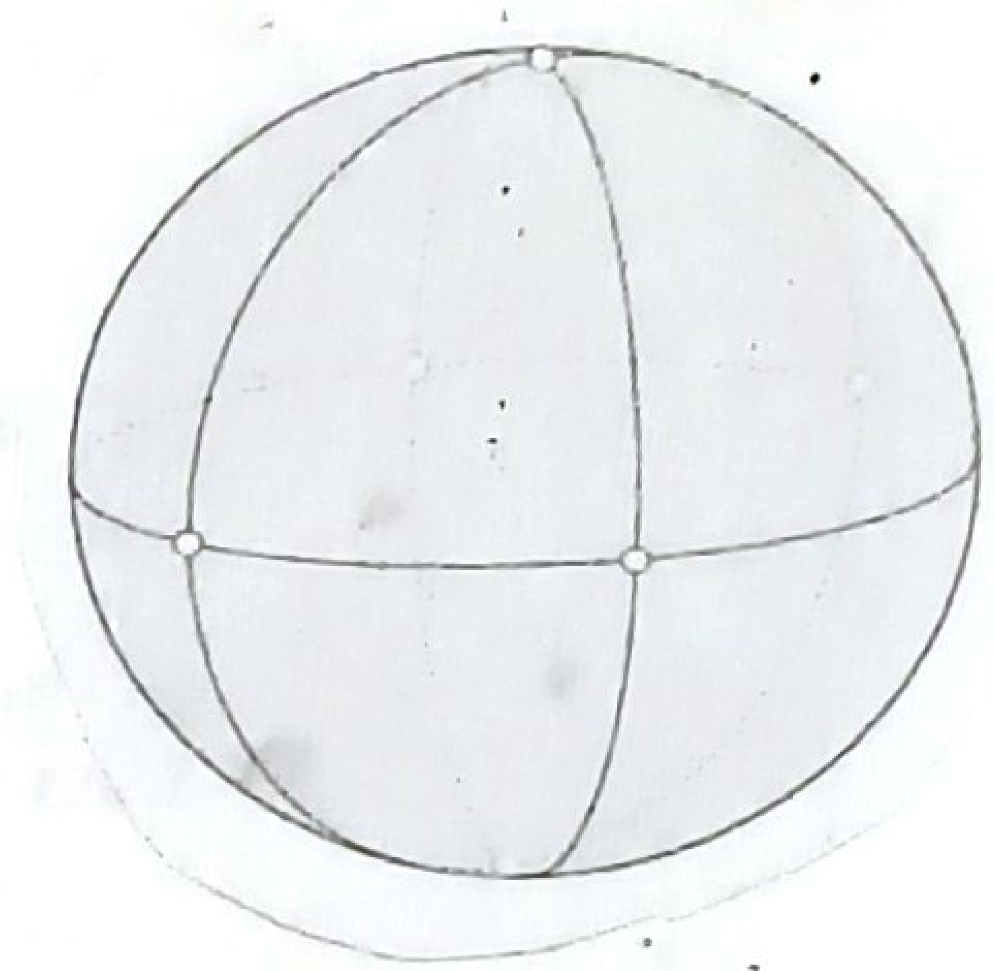
$V - E + F = 6 - 12 + 8 = 2$

$\Rightarrow \chi(S^2) = 2$

$\Rightarrow \boxed{\chi(S^2) = \chi(K)}$



$\cong$



$|K| \cong S^2$

(polyhedron) (sphere)

#1

Simplicial Chain complex and homology

Let K be a simplicial complex with a fixed orientation.

Let S_q denote the set of all oriented q-simplexes of K.

Since each q-simplex, $q \geq 1$, can be oriented in exactly two distinct ways, the number of element in the set S_q is twice the number of q-simplexes in K, whereas the number of oriented 0-simplexes is the same as the number of vertices of K because 0-simplexes are assumed to be always positively oriented.

q - chain of K

let $0 \leq q \leq \dim K$ and $\mathbb{Z}$ be the additive group of integers.

Any map $f : S_q \longrightarrow \mathbb{Z}$ with the property that if $q \geq 1$, then

$$f(-\sigma^q) = -f(\sigma^q) \quad \text{for each } \sigma^q \in S_q$$

is called a q-chain of K.

Note: S_q denote the family of oriented q - simplexes or set of all oriented q - simplexes of K.

* q - chain of K is a function f.

* A 0 - chain is just a mapping form the set of all 0 - simplexes of K to $\mathbb{Z}$.

* $-\sigma^q$ and σ^q are opposite orientations of the same simplex.

The set of all q-chains of K is denoted by $C_q(K)$.

If $q<0$ or $q>\dim K$ we define $C_q(K)=0$

The set $C_q(K)$ of all q-chains is an abelian group with point wise operation viz

① $(f+g)(\sigma^q) = f(\sigma^q) + g(\sigma^q)$ for all

all $\sigma^q \in S_q$

② The zero map $0: S_q \longrightarrow \mathbb{Z}$ defined by

$0(\sigma^q)=0$ for all q- simplexes.

Zero map is the additive identity

i.e (0- chain of K is the additive identity.

③ The additive inverse of f (q-chain of K)

is $-f$

The group $C_q(K)$ is called the q-dimensional chain group of K.

If we imagine the collection of all chain groups $C_q(K)$ arranged in descending order, we have an infinite sequence of abelian group

$$\cancel{C(K)}: \quad \ldots \; 0, C_n(K), C_{n-1}(K), \ldots, C_q(K), \ldots \ldots C_0(K), 0$$

where all $C_q(K)$ are zero except for $q = 0, 1, 2, \ldots, n = \dim(K)$

Simplicial homology groups :-

Suppose that K is a simplicial complex.

For $r \in \mathbb{Z}$ the r-chain group of K, denoted $C_r(K)$, is the free abelian group generated by K_r, the set of (non-empty) oriented r-simplices of K subject to the relation $\sigma + \tau = 0$ i.e $\sigma = -\tau$ whenever σ and τ are the same simplex with the opposite orientations.

An element of this group is called an r-dimensional chain of K.

Thus, for $r \geq 0$, $C_r(K) \cong \mathbb{Z}^k$ where k is the ~~...~~ number of r-simplices in K.

For $r < 0$, $C_r(K) = 0$

An r-chain of K is an integral linear combination $\lambda_1 \sigma_1 + \lambda_2 \sigma_2 + \dots + \lambda_q \sigma_q$ where $\lambda_i \in \mathbb{Z}$ and $\sigma_i \in K_r$.

Define the q- chain group, denoted $C_q(K)$, to be the group whose elements are formal sums

$c_1\sigma_1 + \cdots + c_m\sigma_m$ with $c_i \in \mathbb{Z}$

and where addition in $C_q(K)$ is defined by treating the σ_i as variables.

For example: $(2\sigma_1 + 3\sigma_2) + (4\sigma_1)$

$= 6\sigma_1 + 3\sigma_2$

This makes $C_q(K)$ into a free abelian group whose generators are $\sigma_1, \sigma_2, \ldots, \sigma_m$.

of course, $C_q(K)$ is isomorphic to $\mathbb{Z}^m$.

An element of $C_q(K)$ is called a k-dimensional chain or k-chain

Note:- Think generator as a basis for the group $C_q(K)$

$C_q(K)$ is a free abelian group of rank I_q

$$C_q(K) \cong \underbrace{\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots\cdots \oplus \mathbb{Z}}_{I_q}$$

<u>Note</u>:- $C_q(K)$ is to say that it is the $\mathbb{Z}_2$ or $\mathbb{Z}/2$ - vector space spanned by S_q, i.e the set of all linear combinations $\lambda_1\sigma_1 + \cdots + \lambda_k\sigma_k$ where for $1 \le i \le k$, λ_i is element of $\mathbb{Z}_2$ i.e either 0 or 1 and σ_i is an q-simplex in K. such a linear combination is called an q-chain

For each positively oriented q-simplex σ^q, define a q chain (f_q)

f_q ~~is~~ ~~$f_q f_\tau$~~

$$f_q(\tau^q) = \begin{cases} +1 & \text{if } \tau^q = \sigma^q \\ -1 & \text{if } \tau^q = -\sigma^q \\ 0 & \text{if } \sigma^q \neq \tau^q \end{cases}$$

Then f_q is ~~inded~~ indeed a q-chain and is called an elementary q-chain of K.

Theorem :- For each $q \geq 0$, $C_q(K) = \bigoplus \mathbb{Z} \cdot f_q(\sigma^q)$

where $\sigma^q \in S_q$

The elementary q-chain f_q depends on the q-simplex σ^q and its orientation.

If we fix an orientation for each q-simplex σ^q, then the group

$$C_q(K) = \bigoplus_{\sigma^q \in S_q} \mathbb{Z}.f_q$$ is isomorphic

to the group $\bigoplus \mathbb{Z}.\sigma^q$, where $\mathbb{Z}.\sigma^q$ is the infinite cyclic group generated by the simplex σ^q.

Boundary homomorphism :-

For each q, $0 < q \leq \dim K$, we now define a homomorphism $\partial_q : C_q(K) \longrightarrow C_{q-1}(K)$. Called the boundary homomorphism as follows : on generator σ^q of $C_q(K)$, we define

$$\partial_q(\sigma^q) = \sum_{i=0}^{q} [\sigma^q, \sigma_i^{q-1}] \, \sigma_i^{q-1} \quad \text{—} \; \text{(I)}$$

where σ_i^{q-1} runs over all positively oriented $(q-1)$ faces of σ^q.

Then we extend it over $C_q(K)$ by linearity, i.e. we set

$$\partial_q\left(\sum n_q \sigma^q\right) = \sum n_q \, \partial_q(\sigma^q),$$

where $\partial_q(\sigma^q)$ ~~is~~ defined by (I).

For $q \leq 0$ or $q > \dim K$, we define ∂_q to be the zero homomorphism.

Let us observe that if $\sigma^q = \langle v_0, v_1, \dots, v_q \rangle$ is oriented by the ordering $v_0 < v_1 < \dots < v_q$.

Then σ^q has $(q+1)$ faces ~~$\langle v_0, v$~~

$\langle v_0, v_1, \dots, \hat{v_i}, \dots, v_q \rangle$, $i = 0, 1, \dots, q$

where $\hat{v_i}$ means the vertex v_i has omitted from $\langle v_0, v_1, \dots, v_i, \dots, v_q \rangle$

and each of these faces is oriented by the induced ordering.

Evidently the incidence number $[\sigma^q, \sigma_i^{q-1}]$ is $(-1)^i$ and

Therefore,

$$\partial_q(\sigma^q) = \sum_{i=0}^{q} [\sigma^q, \sigma_i^{q-1}] \sigma_i^{q-1}$$

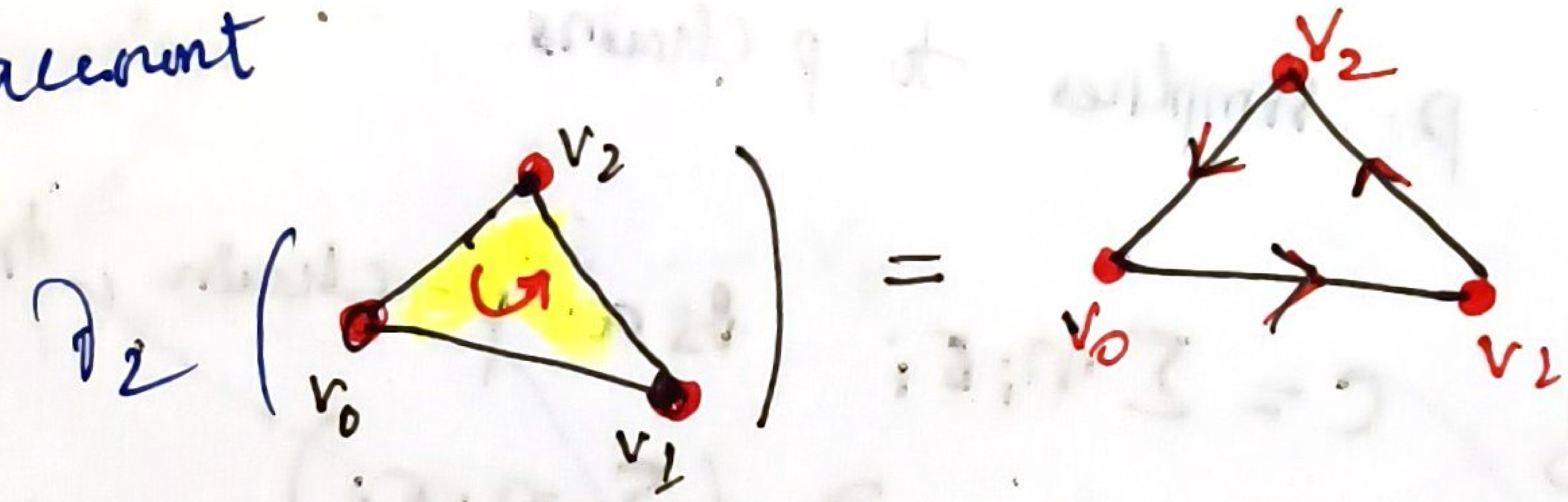
$$\boxed{\partial_q(\sigma^q) = \sum_{i=0}^{q} (-1)^i \langle v_0, v_1, \dots, \hat{v_i}, \dots, v_q \rangle}$$

we define a homomorphism

$$\partial_p : C_p(K) \longrightarrow C_{p-1}(K)$$ called

the boundary operator (p-boundary) or boundary homomorphism.

~~If~~ Intuitively, the boundary of a triangle is made of its three edges. But ~~now~~ now we take the orientation also into account.



we define the homomorphism

$$\partial_p : C_p(K) \longrightarrow C_{p-1}(K)$$ called the

boundary operator as follows:

If $\sigma = [v_0, \dots, v_p]$, $p > 0$ then

$$\partial_p \sigma = \partial_p [v_0, \dots, v_p]$$

$$= \sum_{i=0}^{p} (-1)^i [v_0, \dots, \hat{v_i}, \dots, v_p]$$

where $\hat{v_i}$ means vertex v_i is deleted

from $[v_0, \ldots, v_p]$.

As $C_p(K)$ is trivial for $p < 0$,

∂_p is the trivial homomorphism for $p \leq 0$.

Since ∂_p is a homomorphism, we naturally extend the definition of boundary from

p-simplices to p chains.

If $c = \sum n_i \sigma_i$ is a p-chain, then

$$\partial_p c = \partial_p \left(\sum n_i \sigma_i \right)$$

$$= \sum n_i (\partial_p \sigma_i)$$

If $c = \sum n_i \sigma_i$ is a p-chain, then

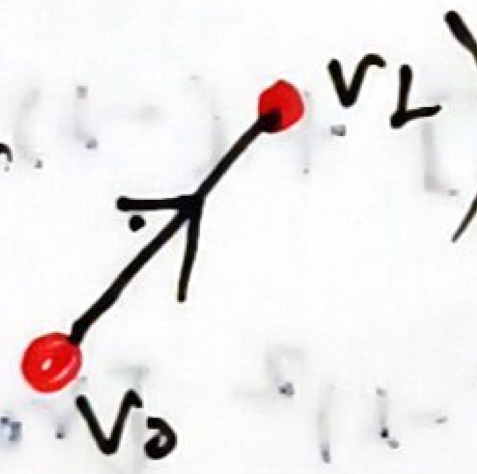
$$\partial_p c = \partial_p \left(\sum n_i \sigma_i\right)$$

$$= \sum n_i (\partial_p \sigma_i)$$

<u>Example:</u>

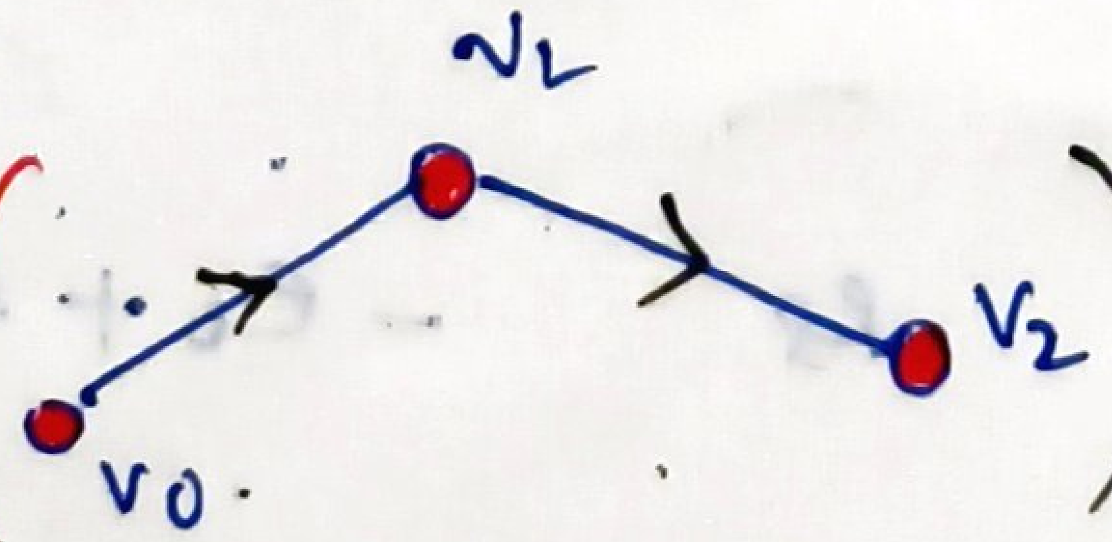
1 - simplex

$$\partial_1 [v_0 v_1] = v_1 - v_0$$

$\partial_1 \Big($ 

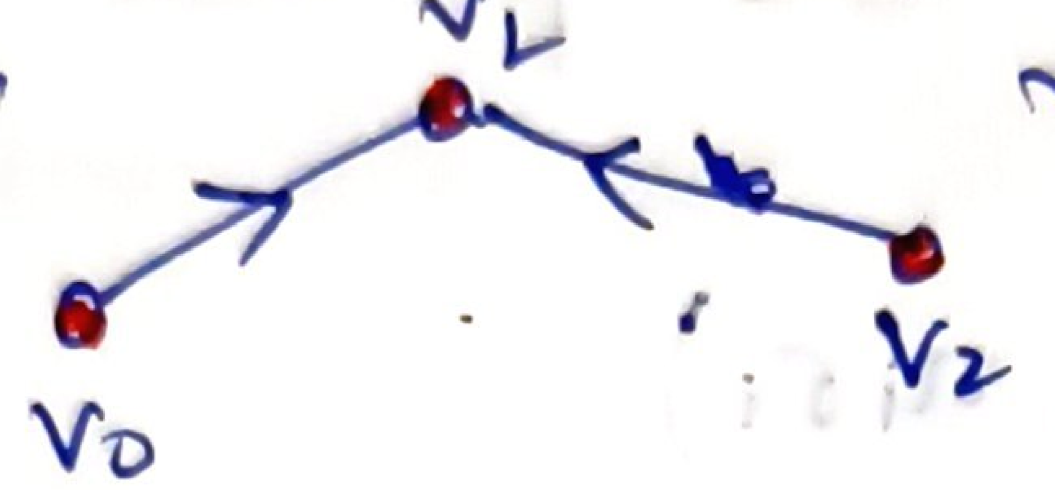
$\Big) = v_1 - v_0$

$$\partial_1 [v_1 v_0] = v_0 - v_1$$

$\partial_1 \Big($ 

$\Big) = v_1 - v_0 + v_2 - v_1$

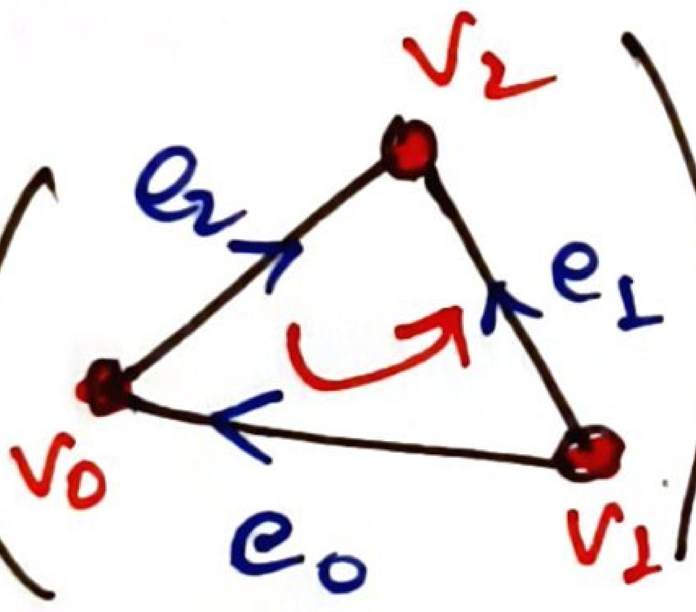
$$= v_2 - v_0$$

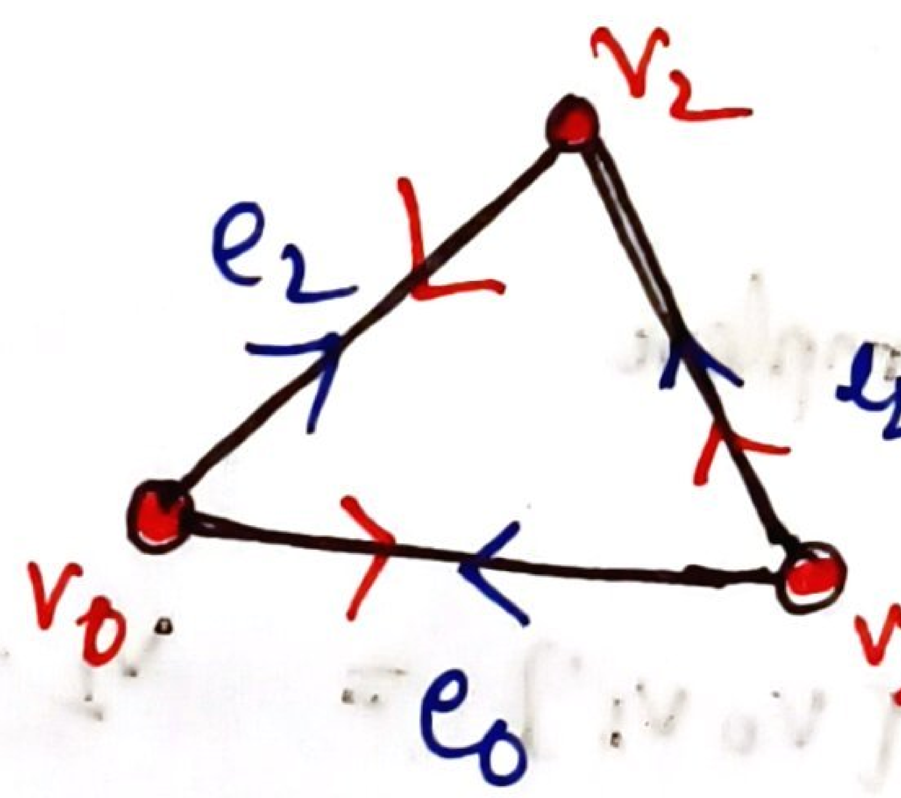
$\partial_1 \Big($ 

$\Big) = 2v_1 - v_0 - v_2$

$= v_1 - v_0 + v_1 - v_2$

2 simplex

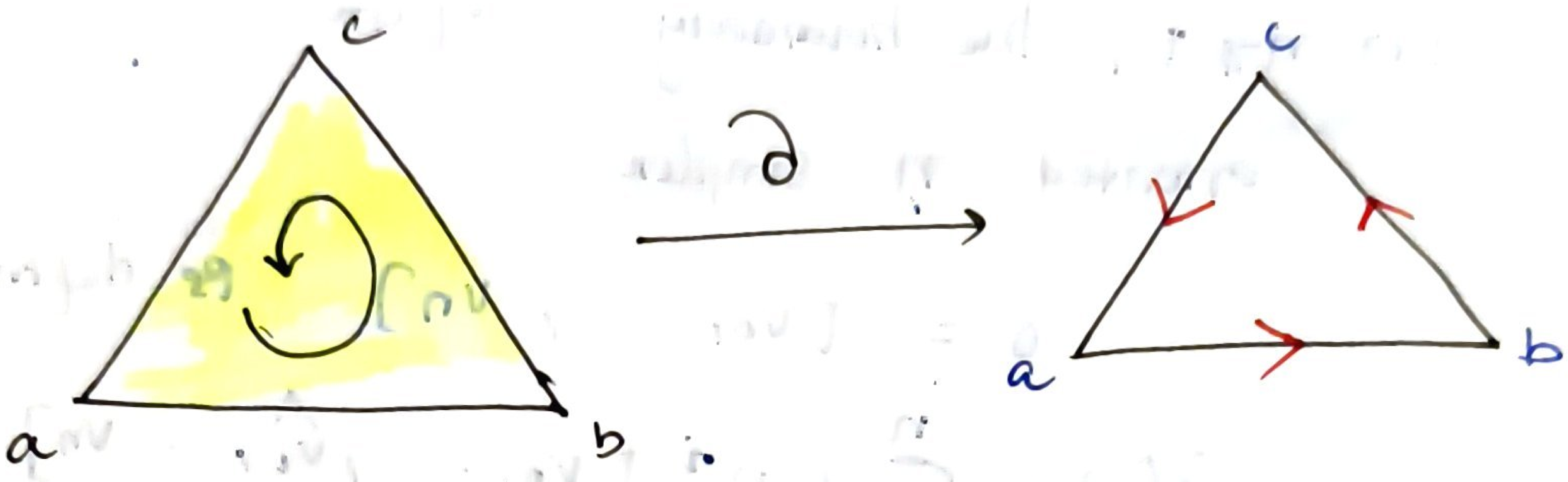
$\partial_2 \Big($ 

$\Big) =$ 

$$\partial_2 [v_0 v_1 v_2] = (-1)^0 [v_1 v_2] + (-1)^1 [v_0 v_2] + (-1)^2 [v_0 v_1]$$

$$= [v_1 v_2] - [v_0 v_2] + [v_0 v_1]$$

The 1-boundary $\quad - e_0 + e_1 - e_2$



$$\partial[a,b,c] = (-1)^0[\hat{a},b,c] + (-1)^1[a,\hat{b},c] + (-1)^2[a,b,\hat{c}]$$

$$= [b,c] - [a,c] + [a,b]$$

$$= [b,c] + [c,a] + [a,b]$$

For $n \geq 1$, the boundary of an oriented n- simplex

$\sigma = [v_0, \ldots, v_n]$ is defined by

$$\partial(\sigma) = \sum_{i=0}^{n} (-1)^i [v_0, \ldots, \hat{v_i}, \ldots v_n]$$

$[v_0, v_1, v_2, \ldots, \hat{v_i}, \ldots v_n]$ represents the simplex with v_i removed.

The boundary of σ can be viewed as the sum of the $(n-1)$ dimensional face of σ each of which the induced orientation

$(-1)^i [v_0, \ldots, \hat{v_i}, \ldots v_n]$.

The boundary of an n-chain $\sum_{i=1}^{k} c_i \sigma_i$ is an $(n-1)$ chain:

$$\partial\left(\sum_{i=1}^{k} c_i \sigma_i\right) = \sum_{i=1}^{k} c_i \, \partial(\sigma_i)$$

Thus the boundary operator is a homomorphism

$\partial_n : C_n(K) \longrightarrow C_{n-1}(K)$ for each $n \geq 0$

lemma : For each q, the composite homomorphism $\partial_{q-1} \circ \partial_q : C_q(K) \longrightarrow C_{q-2}(K)$ is the zero map.

(#) Kernel of the boundary homomorphism :

Let K be an oriented complex.

A q-chain $z_q \in C_q(K)$ is called a q-dimensional cycle of K (or just a q-cycle) if $\partial_q(z_q) = 0$.

The set of all q-cycles of K, denoted by $Z_q(K)$, is the kernel of the boundary homomorphism $\partial_q : C_q(K) \longrightarrow C_{q-1}(K)$ and, therefore, is a subgroup of $C_q(K)$.

Thus $Z_q(K)$ is referred to as group of q-cycles of K.

Given a simplicial complex K, the kernel of the boundary homomorphism $\partial_q : C_q(K) \longrightarrow C_{q-1}(K)$ is called the q-cycle group of K and is denoted $Z_q(K)$. Thus

$$\boxed{Z_q(K) = \{ x \in C_q(K) \mid \partial_q(x) = 0 \}}$$

An element of $Z_q(K)$ is called an r-dimensional (or just an $r$cycle) of K.

Group of q-boundaries of K

An element $b_q \in C_q(K)$ is said to be a q-dimensional boundary (or just a q-boundary) of K if there exist a ~~K ∈ Cq~~ $m \in C_{q+1}(K)$ such that

$$\partial_{q+1}(m) = b_q.$$

The set of all q-boundaries, being the homomorphic image $\partial_{q+1}(C_{q+1}(K))$ is also a subgroup of $C_q(K)$. It is denoted by $B_q(K)$ and is referred to as group of q-boundaries of K.

Given a simplicial complex K, the image of the boundary homomorphism

$$\partial_{q+1} : C_{q+1}(K) \longrightarrow C_q(K) \text{ is}$$

called the ~~q-boundary~~ q-boundary group of K and is denoted $B_q(K)$.

Thus $B_q(K) = \{x \in C_q(K) \mid x = \partial_{q+1}(y)$ for some $y \in C_{q+1}(K)\}$

$$B_q(K) = \{x \in C_q(K) \mid x = \partial_{q+1} \text{ for some } y \in C_{q+1}(K)\}.$$

An element of $B_q(K)$ is called an q-dimensional boundary (or just an q-boundary) of K.

The chain complex is the sequence of chain groups connected by boundary homomorphisms,

$$\cdots \xrightarrow{\partial_{p+2}} C_{p+1} \xrightarrow{\partial_{p+1}} C_p \xrightarrow{\partial_{p-1}} C_{p-1}$$

$$\cdots \xrightarrow{\partial_{p+2}} C_{p+1} \xrightarrow{\partial_{p+1}} C_p \xrightarrow{\partial_p} C_{p-1} \xrightarrow{\partial_{p-1}} \cdots$$

If $\dim K = n$, then there is sequence

$$C(K): \cdots 0 \longrightarrow C_n(K) \xrightarrow{\partial_n} \cdots \rightarrow C_{q+1}(K) \xrightarrow{\partial_{q+1}} C_q(K) \xrightarrow{\partial_q} C_{q-1}(K) \rightarrow \cdots \longrightarrow C_0(K) \longrightarrow 0 \longrightarrow \cdots$$

of free abelian groups and groups homomorphisms in which the composite of any two consecutive homomorphism is 0.

This long sequence is called the oriented simplicial chain complex of K.

~~So~~ Since $\partial_q \circ \partial_{q+1} = 0$ for each q,

$\text{Im}\, \partial_{q+1} \subseteq \ker \partial_q$

$\Rightarrow B_q(K) \subseteq Z_q(K)$

Hence $\boxed{B_q(K) \subset Z_q(K) \subset C_p(K)}$

Chain complexes :-

A sequence C of Abelian groups and their homomorphisms

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots$$

infinite in both directions, is called a chain complex if for all n we have the equality $\partial_n \partial_{n+1} = 0$.